Electrohydrodynamics: equations and their application to paradigmatic flows

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Outline

Outline

Physical system

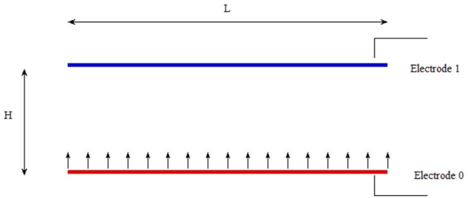


Figure: Geometrical configuration

Physical system

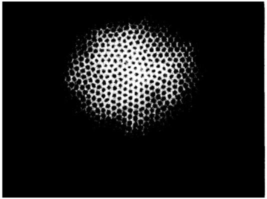


Figure: Atten and Lacroix 1978

Equations

An incompressible dielectric liquid of negligible conductivity.

$$
\begin{aligned}
\nabla^2 \Phi &= -\frac{q}{\epsilon} \\
\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{j} &= 0 \\
\nabla \cdot \mathbf{u} &= 0 \\
\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p \right) &= \eta \nabla^2 \mathbf{u} + q\mathbf{E}
\end{aligned}
$$

where $\mathbf{j} = q(K\mathbf{E} + \mathbf{u} - D\nabla q)$ and $\mathbf{E} = -\nabla\Phi$.

Boundary conditions

The electric boundary conditions are:

$$\begin{aligned}\Phi(z=0) &= V\\ \Phi(z=d) &= 0\\ q(z=0) &= q_0\end{aligned}$$

The mechanical boundary conditions are

$$\mathbf{u} = 0 \ \text{ at } \ z = 0, d$$

Non-dimensional equations

$$
\begin{aligned}
\nabla^2\Phi &= -q \\
\frac{\partial q}{\partial t} + \nabla \cdot (q(\mathbf{E} + u) - \alpha \nabla q) &= 0 \\
\nabla \cdot \mathbf{u} &= 0 \\
\frac{T}{M^2}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p\right) &= \nabla^2 \mathbf{u} + Tq\mathbf{E}
\end{aligned}
$$

with the boundary conditions:

$$
\begin{aligned}
\Phi(z=0) &= 1 \\
\Phi(z=d) &= 0 \\
q(z=0) &= C \\
\mathbf{u} &= 0 \quad \text{at} \quad z = 0, d
\end{aligned}
$$

Non-dimensional parameters

- $T = \dfrac{\epsilon V}{k\eta}$.
- $M = \sqrt{\epsilon/\rho}/K$. The quantity $\sqrt{\epsilon/\rho}$ is known as the Electrohydrodynamic mobility.
- $C = q_0 d^2/\epsilon V$. Recall that $\epsilon V/d^2$ existing on the electrodes as a consequence of the applied voltage.
- $\alpha = D/KV$. Einstein's relation implies $D/K = k_B\Theta/e \sim 0.025$ V at room temperature. Therefore, $\alpha = 0.025/V$.

Charge and field distribution in the hydrostatic state

$$
\begin{aligned}
E_{0z} &= \sqrt{2j(z+b)} \\
q_0 &= \sqrt{\frac{j}{2(z+b)}}
\end{aligned}
$$

The electric boundary conditions give:

$$
\begin{aligned}
\sqrt{j}((1+b)^{3/2} - b^{3/2}) &= \frac{3}{2\sqrt{2}} \\
2bC^2 &= j
\end{aligned}
$$

Theses two equations provide the current density for every value of C.

The current density is a function of the injection level

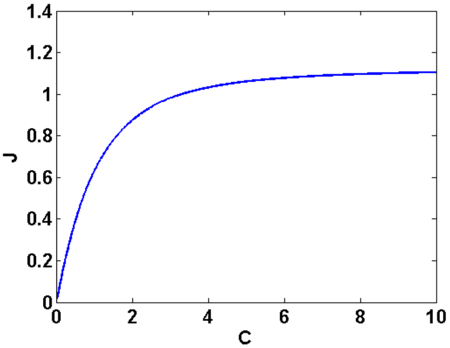


Figure: Current density as a function of injection level.

Weak and strong injection limits

- $C \to \infty$

$$\begin{aligned} E_{0z} &= \frac{3}{2}\sqrt{z} \\ q_0 &= \frac{3}{4\sqrt{z}} \\ j &= \frac{9}{8} \end{aligned}$$

- $C << 1$

$$\begin{aligned} E_{0z} &= 1 - \frac{C}{2} + Cz + O(C^2) \\ q_0 &= C - C^2 z + O(C^3) \\ j &= C + O(C^2) \end{aligned}$$

Outline

Perturbation of the hydrostatic state

$$
\begin{aligned}
\Phi &= \Phi_0 + \delta\phi \\
q &= Q_0 + \delta q \\
p &= p_0 + \delta p \\
& \quad \mathbf{u}
\end{aligned}
$$

Neglecting non-linear terms yields:

$$
\begin{aligned}
\nabla^2 \delta\phi &= -\delta q \\
\frac{\partial \delta q}{\partial t} + \nabla \cdot (\delta q \mathbf{E}_0 + Q_0 \delta \mathbf{E} + Q_0 \mathbf{u}) &= 0 \\
\nabla \cdot \mathbf{u} &= 0 \\
\frac{T}{M^2}\left(\frac{\partial \mathbf{u}}{\partial t} + \nabla \delta p\right) &= \nabla^2 \mathbf{u} + T(Q_0 \delta \mathbf{E} + \delta q \mathbf{E}_0)
\end{aligned}
$$

Perturbation of the hydrostatic state

Eliminating the pressure term by taking twice the curl of equation (33). Only the z-component of the velocity is needed, which is:

$$\frac{T}{M^2}\frac{\partial \nabla^2 u_z}{\partial t} = \nabla^4 u_z + T\nabla_s^2\left(\frac{d\Phi_0}{dz}\nabla^2\delta\phi - \frac{d^3\Phi_0}{dz^3}\delta\phi\right)$$

Normal modes

Introducing normal modes:

$$\begin{aligned} u_z &= u(z)e^{i(k_x x + k_y y)}e^{st} \\ \delta\phi &= g(z)e^{i(k_x x + k_y y)}e^{st} \end{aligned}$$

the equations become:

$$\frac{T}{M^2}s(\frac{d^2u}{dz^2}-k^2u) = (\frac{d^2}{dz^2}-k^2)^2u-k^2T(\frac{dQ_0}{dz}g-E_0(\frac{d^2}{dz^2}-k^2)g)$$

$$-s(\frac{d^2g}{dz^2}-k^2g)-\frac{dQ_o}{dz}\frac{dg}{dz}-2Q_0(\frac{d^2g}{dz^2}-k^2g)- E_0\frac{d}{dz}(\frac{d^2g}{dz^2}-k^2g)+\frac{dQ_0}{dz}u=0$$

Boundary conditions

The boundary conditions are expressed in terms of u and g in the following way:

$$\begin{aligned} u(0) &= 0 \\ \frac{du}{dz}(0) &= 0 \\ u(1) &= 0 \\ \frac{du}{dz}(1) &= 0 \\ g(0) &= 0 \\ g(1) &= 1 \\ \frac{d^2g}{dz^2}(0) &= 0 \end{aligned}$$

Analytical approximative solution for weak injection

The equations simplify for $C << 1$ and $(s = 0)$.
With $E_0 = 1$ and $Q_0 = C - C^2 z$.

$$\begin{aligned} \left(\frac{d^2}{dz^2} - k^2\right)^2 u &= k^2 T \delta\overline{q} \\ \frac{d}{dz}\delta\overline{q} - C^2 u &= 0 \end{aligned}$$

where $\delta\overline{q} = -(d^2 g/dz^2 - k^2 g)$.

The second equation reveals the instability mechanism.

Galerkin method

Propose a function:

$$u(z) = z^2(1-z)^2$$

that fulfills the boundary conditions for u.
The second equation gives:

$$\delta\overline{q} = C^2\left(\frac{z^3}{3} - \frac{z^4}{2} + \frac{z^5}{5}\right)$$

Multiplying the first equation by u and integrating between 0 and 1:

$$\int_0^1 u\left(\frac{d^2}{dz^2} - k^2\right)^2 u\, dz = k^2 T \int_0^1 u\delta\overline{q}\, dz$$

Galerkin method

For $u(z) = z^2(1-z)^2$ this equation becomes:

$$TC^2 = \frac{20}{7}k^2 + \frac{120}{7} + \frac{1440}{k^2}$$

This function has a minimum at $k = 4.74$ for which $TC^2 = 197$.

The exact values, determined by numerical methods, are $k = 4.57$ and $TC^2 = 221$.

Numerical solution

We have a seventh order set of homogenous ordinary linear differential equations with non-constant coefficients. This problem was solved by Atten and Moreau by advanced mathematical methods.

Nowadays it can be efficiently solved by numerical algorithms commercially available. I have used the function bvp4c, available through version 6.1 of Matlab

The resolution provides T as a function of k

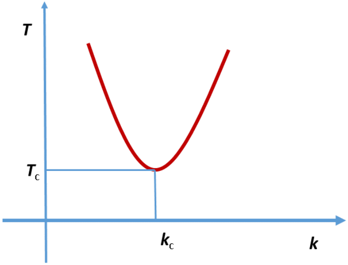


Figure: T as a function of k.

For every value of C there is a minimum value of T

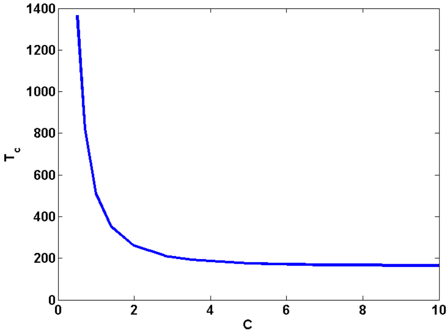


Figure: Linear stability threshold T_c as a function of injection level C.

For strong injection the instability appears at given voltage.

Outline

The charge conservation equation is a first order hyperbolic equation

$$\frac{\partial q}{\partial t} + \nabla \cdot (q(\mathbf{E} + u)) = 0$$

Taking into account that $\nabla \cdot \mathbf{E} = q$ and $\nabla \cdot \mathbf{u} = 0$ it can be written:

$$\frac{\partial q}{\partial t} + (\mathbf{E} + u) \cdot \nabla q = -q^2$$

This is a first order partial differential equation.

It is equivalent to the set of ordinary differential equations:

$$
\begin{aligned}
\frac{dq}{dt} &= -q^2 \\
\frac{dx}{dt} &= u_x + E_x \\
\frac{dz}{dt} &= u_z + E_z
\end{aligned}
$$

The solution of the first equation is

$$q(t) = \frac{q_0}{1 + q_0 t}$$

The value of q at a given point is determined by the characteristic line connecting that point to the boundary.

Numerical diffusion

The origin of numerical diffusion can be understood from a simplify analysis. Consider the one dimensional equation:

$$\frac{\partial q}{\partial t} + \frac{\partial uq}{\partial x} = 0$$

Assuming u constant and $u > 0$, typical finite-difference *upwind* discretization of this equation in a mesh $\{x_i\}_{i=1,ldots,m}$ is:

$$q_i^{t+\Delta t} = q_i^t - \frac{u\delta t}{\delta x}(q_i^t - q_{i-1}^t)$$

A power expansion gives:

$$\begin{aligned} q_i^{t+\Delta t} &= q_i^t + \frac{\partial q_i}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 q_i}{\partial t^2}(\Delta t)^2 + O(\Delta^3 t) \\ q_{i-1}^t &= q_i^t - \frac{\partial q_i}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 q_i}{\partial x^2}(\Delta x)^2 + O(\Delta^3 x) \end{aligned}$$

Taking $\partial/\partial t$ in the original equation, with u constant, is

$$\frac{\partial^2 q}{\partial t^2} + u\frac{\partial^2 q}{\partial x \partial t} = 0$$

Collecting the terms we get:

$$\frac{\partial q}{\partial t} + \frac{\partial uq}{\partial x} = \alpha_N \frac{\partial^2 q}{\partial x^2} + O(\Delta^2 t, \Delta^2 x)$$

with

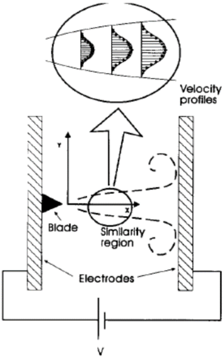
$$\alpha_N = \frac{1}{2} u \Delta x (1 - \frac{u \Delta t}{\Delta x})$$

The numerical errors behave diffusively.

For $\Delta x \sim 1/100$, $u \sim 1$ and $\Delta t \sim 1/200$ is $\alpha_N \sim 2.5 \times 10^{-3}$.

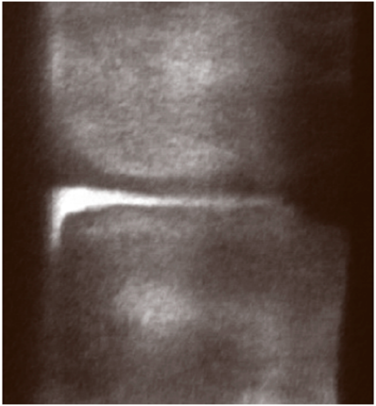
Outline

EHD plumes



Physical system

EHD plumes



Schlieren image

- The electric field is consider to be constant $\mathbf{E} = E_0\mathbf{e}_z = (V/d)\mathbf{e}_z$.
- Charge diffusion and drift are neglected ($u >> KE$).
- The structure of the flow is that of a boundary layer flow. In this case this implies neglecting $\partial^2 u/\partial z^2$ in front of $\partial^2 u/\partial x^2$.
- The flow is steady.
- The pressure is constant. This will not be the case if recirculation or wall effects are not negligible.

Outline

Equations

$$
\begin{aligned}
\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} &= 0 \\
u_x \frac{\partial u_x}{\partial x} + u_z \frac{\partial u_z}{\partial z} &= \nu \frac{\partial^2 u_x}{\partial x^2} + \frac{1}{\rho} q E_z \\
u_x \frac{\partial q}{\partial x} + u_z \frac{\partial q}{\partial z} &= 0
\end{aligned}
$$

Boundary conditions

$$u_x = \frac{\partial u_z}{\partial x} = \frac{\partial q}{\partial x} = 0 \qquad \text{at} \;\; x = 0$$
$$u_x = q = 0 \qquad \text{for} \;\; x \to \infty$$

The electric current through any section is invariant:

$$J = \int_{-\infty}^{\infty} q u_z \, dx$$

To fix J is equivalent to assign a level of injection to the injector.

Similarity method

Basic idea

- If they are conveniently scaled, the velocity and charge profile have the same form at every section.

Using the stream function $\Psi(x, z)$:

$$u_x = -\partial\Psi/\partial z \quad u_z = \partial\Psi/\partial x$$

We look for a solution of the type:

$$\eta = c(z)x, \quad \Psi = \nu d(z) f(\eta), \quad q = a(z) g(\eta)$$

.

Similarity method

Chosing the constants appropriately

$$
\begin{aligned}
\eta &= \left(\frac{JE}{16\rho\nu^3}\right)^{1/5} \frac{x}{z^{2/5}} \\
\Psi &= 4\nu \left(\frac{JE}{16\rho\nu^3}\right)^{1/5} z^{3/5} f(\eta) \\
q &= \left(\frac{\rho J^4}{64\nu^2 E}\right) z^{-3/5} g(\eta)
\end{aligned}
$$

The underlying idea of the method is that as long as the flow is open there is not a typical length scale.

The profiles of velocity and charge are given by the solution of

$$
\begin{aligned}
f''' + \frac{12}{5} f f'' - \frac{4}{5} f'^2 - g &= 0 \\
f' g + f g' &= 0
\end{aligned}
$$

along with the boundary conditions

$$
\begin{aligned}
f = f' = g' = 0 \quad &\text{at} \quad \eta = 0 \\
f' = 0 \quad &\text{for} \quad \eta \to \infty \\
\int_{-\infty}^{\infty} f' g \, d\eta &= 1
\end{aligned}
$$

The second equation and the boundary conditions imply that

$$
g = 0
$$

except at $\eta = 0$

Singularity at the origin

The charge profile can be expressed:

$$g(\eta) = g_0 \delta(\eta)$$

Integrating between 0 and 0^+

$$f''' + \frac{12}{5} f f'' - \frac{4}{5} f'^2 - g = 0$$

gives a new boundary condition

$$f'(0^+) f''(0^+) + \frac{1}{2} = 0$$

Equivalent problem

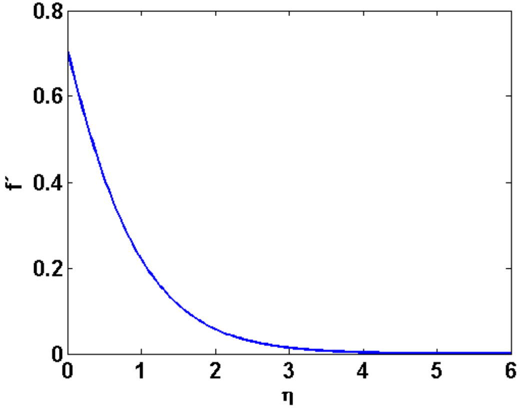
The profile of velocity is solution of

$$f''' + \frac{12}{5} f f'' - \frac{4}{5} f'^2 = 0$$

with boundary conditions

$$\begin{aligned} f'(0^+) f''(0^+) + \frac{1}{2} &= 0 \\ f(0^+) &= 0 \\ f'(\infty) &= 0 \end{aligned}$$

Velocity profile



Velocity profile of a 2D EHD plume

Comparison with a thermal plume

For a thermal plume the second equation is

$$g'' + \frac{12}{5}\mathrm{Pr}(f'g + fg') = 0$$

whose solution is

$$g(\eta) = g(0)\exp - \left(\frac{12}{5}\mathrm{Pr}\int_0^\eta f(s)\,ds\right)$$

Conclusion

- Thermal plumes in the limit of infinite Prandtl number are similar to EHD plumes.
- The thickness of the charge region plays the role of the thickness of the thermal layer.

Outline

Equations

$$
\begin{aligned}
\frac{\partial r u_z}{\partial z} + \frac{\partial r u_r}{\partial r} &= 0 \\
u_z \frac{\partial u_z}{\partial z} + u_r \frac{\partial u_r}{\partial r} &= \frac{\nu}{r} \frac{\partial}{\partial r}(r \frac{\partial u_z}{\partial r}) + \frac{1}{\rho} q E_z \\
u_r \frac{\partial q}{\partial r} + u_z \frac{\partial q}{\partial z} &= 0
\end{aligned}
$$

Boundary conditions

$$u_r = \frac{\partial u_z}{\partial r} = \frac{\partial q}{\partial r} = 0 \qquad \text{at} \;\; r = 0$$
$$u_z = q = 0 \qquad \text{for} \;\; r \to \infty$$

To these conditions the constancy of the electric current on any section of the flow must be added:

$$J = 2\pi \int_0^\infty q u_z r \, dr$$

Similarity method

Introducing the stream function Ψ:

$$u_r = -\frac{1}{r}\partial\Psi/\partial z \quad u_z = \frac{1}{r}\partial\Psi/\partial r$$

Following a similar procedure gives

$$\begin{aligned} \eta &= \left(\frac{JE}{2\pi\rho\nu^3}\right)^{1/4} \frac{r}{z^{1/2}} \\ \Psi &= 4\nu z f(\eta) \\ q &= \frac{J}{2\pi\nu z} g(\eta) \end{aligned}$$

The exponents of the scaling laws are different

- $\eta \sim x/z^{2/5}$ for 2D plumes
- $\eta \sim r/z^{1/2}$ for axisymmetric plumes

The profiles of velocity and charge are given by the solution of

$$\begin{aligned} \frac{f'''}{\eta} + \frac{f-1}{\eta}\left(\frac{f'}{\eta}\right) + g &= 0 \\ f'g + fg' &= 0 \end{aligned}$$

along with the boundary conditions

$$\begin{aligned} (\frac{f'}{\eta})' = \frac{f}{\eta} - \frac{1}{2}f'' = g' = 0 \quad \text{at} \quad \eta = 0 \\ \frac{f'}{\eta} = g = 0 \quad \text{for} \quad \eta \to \infty \\ \int_0^\infty f'g\,d\eta \quad = \quad 1 \end{aligned}$$

Singularity at the origin

- A delta function does not give a finite velocity.
- In the thermal problem the velocity at the axis diverges as

$$\frac{f'}{\eta}(\eta = 0) \simeq \sqrt{\ln \mathrm{Pr}}$$

Therefore, for axisymmetric EHD plumes it is

$$\frac{f'}{\eta}(\eta = 0) \simeq \sqrt{\ln(-a^2)}$$

with a the thickness of the charged layer in η coordinates.

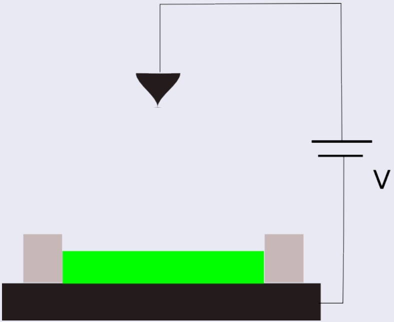
Conclusion

- Unlike the 2D case, the velocity at the axis of axisymmetric plumes depends on the thickness of the charged layer. However, this dependency is very weak

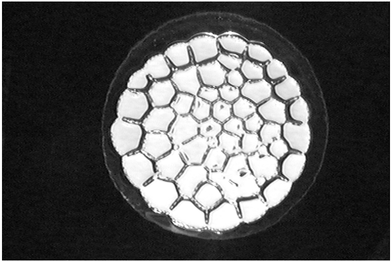
Outline

Introduction

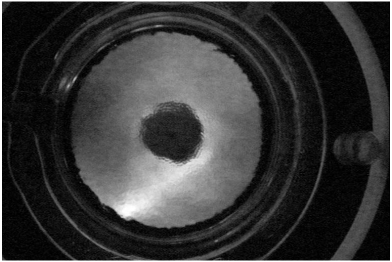
Experimental setup



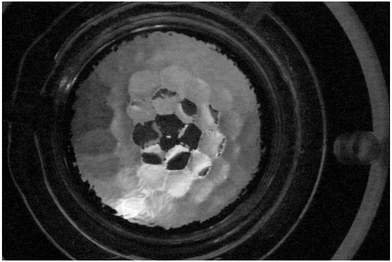
Experimental set up



Olive oil (ohmic) at 10.75 kV



Silicone oil (non ohmic) at 5 kV



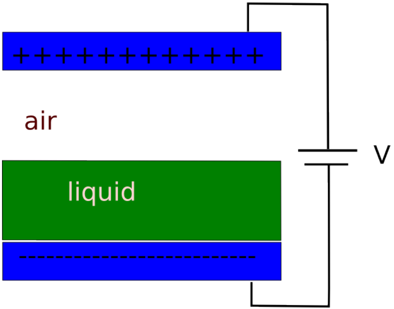
Silicone oil (non ohmic) at 6 kV

Differences

- Ohmic regime $\longrightarrow$ large convection cells (Rose-Window instability)
- Non-ohmic regime $\longrightarrow$ two different patterns coexist

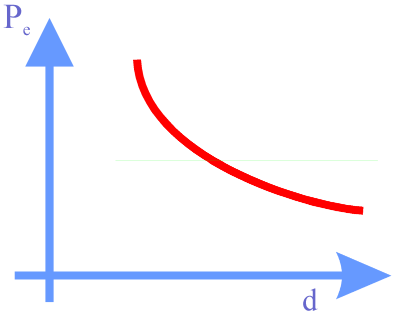
Outline

Instability mechanism for Rose-window pattern



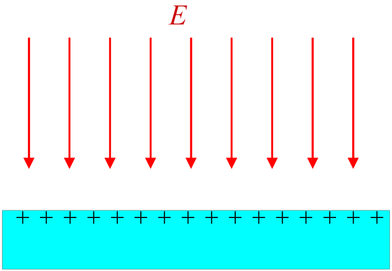
Simplified configuration

Instability mechanism for Rose-window pattern



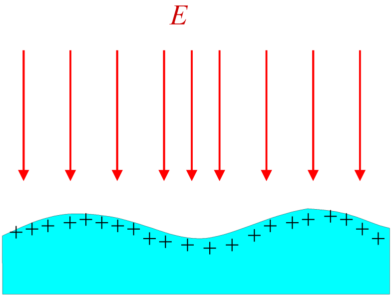
Electric pressure onto the surface
(ohmic behavior)

Instability mechanism for Rose-window pattern



Electric field for an unperturbed interface

Instability mechanism for Rose-window pattern



Electric field (perturbed)

Outline

Equations in the air region

$$\nabla^2 \phi = -\frac{q}{\epsilon_0},$$

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

with $\mathbf{J} = qK_a\mathbf{E}$

$$-\nabla p + \rho_a g \mathbf{e}_z + q\mathbf{E} = 0,$$

Air motion is neglected

Equations for the liquid region (Ohmic regime)

$$\nabla^2\phi = 0,$$

$$\nabla \cdot \mathbf{J} = 0$$

with $\mathbf{J} = \sigma\mathbf{E}$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\rho_l\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho_l g\mathbf{e}_z + \eta\nabla^2\mathbf{u}$$

no electric body force

Boundary conditions at the electrodes

$$
\begin{aligned}
\phi &= V \quad \text{at} \quad z = -L, \\
q &= q_0 \quad \text{at} \quad z = -L, \\
\phi &= 0 \quad \text{at} \quad z = d, \\
\mathbf{u} &= 0 \quad \text{at} \quad z = d.
\end{aligned}
$$

charge injection

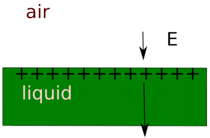
Boundary conditions at the interface (Ohmic regime)

$$[\mathbf{E}] \times \mathbf{n} = 0, \quad [\epsilon \mathbf{E}] \cdot \mathbf{n} = q_s.$$

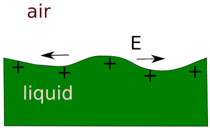
$$\frac{\partial q_s}{\partial t} + \nabla_s \cdot (q_s \mathbf{u}) + [\mathbf{J}] \cdot \mathbf{n} - [q\mathbf{u}] \cdot \mathbf{n} = 0.$$

$$-\frac{\partial f}{\partial t} + u_z + \mathbf{u} \cdot \nabla_s f = 0,$$

Electric stresses



normal stresses



tangential stresses

Continuity

$$\eta\,\mathbf{t}\cdot\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\cdot\mathbf{n}+q_s\mathbf{E}\cdot\mathbf{t}=0,$$

$$-[p]+\eta\,\mathbf{n}\cdot\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\cdot\mathbf{n}+\left[\epsilon(\mathbf{E}\cdot\mathbf{n})^2\right]-\left[\tfrac{1}{2}\epsilon E^2\right]-\gamma\left(\nabla_s\cdot\mathbf{n}\right)=0.$$

Equations and boundary conditions for the non-Ohmic regime

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

where $\mathbf{J} = q(K_l\mathbf{E} + \mathbf{u})$.

$$\rho_l(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \rho_l g \mathbf{e}_z + \eta \nabla^2 \mathbf{u} + q\mathbf{E}.$$

$$[\epsilon \mathbf{E}] \cdot \mathbf{n} = 0.$$

$$[\mathbf{J}] \cdot \mathbf{n} - [q\mathbf{u}] \cdot \mathbf{n} = 0.$$

$$\eta\, \mathbf{t} \cdot \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \cdot \mathbf{n} = 0.$$

Differences between both models

	Ohmic	Non-Ohmic
Electric current	$\mathbf{J} = \sigma \mathbf{E}$	$\mathbf{J} = qK_I\mathbf{E}$
Volume force	0	$q\mathbf{E}$
Surface charge density	$[\epsilon \mathbf{E}] \cdot \mathbf{n} = q_s$	$[\epsilon \mathbf{E}] \cdot \mathbf{n} = 0$
Tangential stress at the interface	$q_s \mathbf{E} \cdot \mathbf{t}$	0

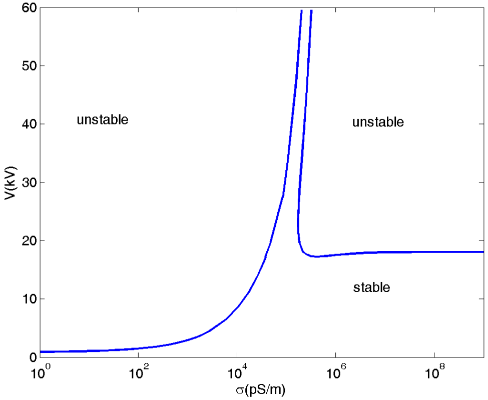
Stability study

$$
\begin{aligned}
\phi &= \phi_s + \delta\phi \\
\delta\phi &= g(z)\exp(\omega t)\exp(i(k_x x + k_y y))
\end{aligned}
$$

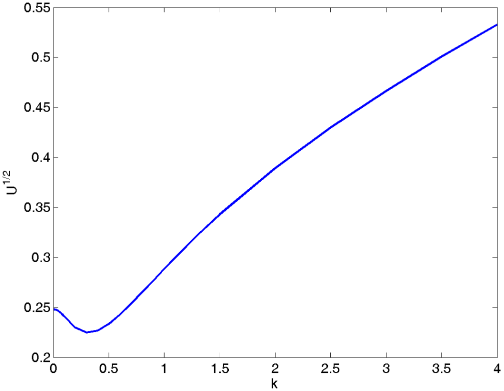
- Ohmic case: set of 9 first order linear differential equations with 10 boundary conditions.
- Non Ohmic case: set of 10 first order linear differential equations with 11 boundary conditions.

Outline

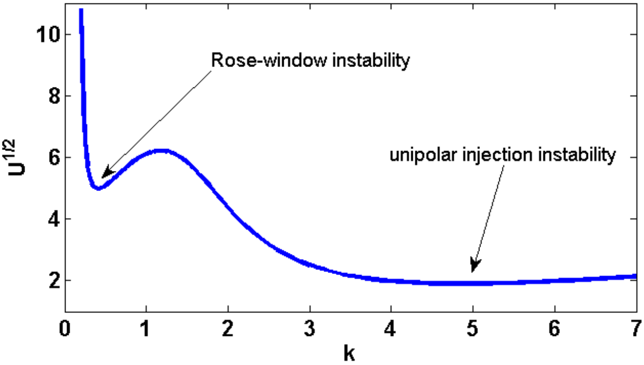
Results



Stability diagram for an Ohmic liquid as a function of the conductivity



Stability parameter $U = \frac{\epsilon_0 \rho_l V^2}{\eta^2}$ as a function of the wavenumber (Ohmic case).



Stability parameter as a function of the wavenumber (non Ohmic case).

Conclusion

- The mathematical equations of EHD flow are well established and tested.
- Semi-analytical methods help to understand the underlying physical mechanisms.
- The numerical resolution of EHD flows require algorithms that avoid numerical diffusion.

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