Antonio Castellanos and Galilean Electromagnetism : A Historical Perspective

$$\frac{\frac{1}{2}\epsilon E^2}{\frac{1}{2}\frac{B^2}{\mu}} = \frac{E^2}{c^2 B^2} = \left(\frac{E}{cB}\right)^2$$

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France

advancing the frontiers

https://en.wikipedia.org/wiki/Antonio_Castellanos_Mata

One of the important events in his life was meeting **Pierre Atten**, who became his close friend and inspired in him a love for electrohydrodynamics.

He worked in various fields of science:

1. **Electrohydrodynamics,**
2. Gas discharges at atmospheric pressure,
3. Cohesive **granular materials.**

Antonio Castellanos Mata
(07.03.1947 – 27.01.2016)

Noteworthy results include:

1. **Galilean limits of electromagnetism.**
2. Temperature equation and entropy production in electrohydrodynamics.
3. Self-similar solution for two-dimensional and axisymmetric plumes.
4. Seminal works on numerical simulation of electrohydrodynamic flows.
5. Physical mechanism of electrothermohydrodynamic instabilities.
6. Energy cascade in electrohydrodynamic turbulence.
7. Stabilization of dielectric liquid bridges by ac electric fields.
8. Theory of AC electroosmosis and electrothermal flows in microsystems.
9. Absence of inertial regimes in fine powders.
10. Model of elastoplastic contact between two powder particles.
11. Microstructure characterization of fluidized bed of fine particles: aggregation, solidlike-fluidlike transition, fluctuations, influence of electromagnetic fields.

IL NUOVO CIMENTO VOL. 14 B, N. 2 11 Aprile 1973

Galilean Electromagnetism.

M. LE BELLAC
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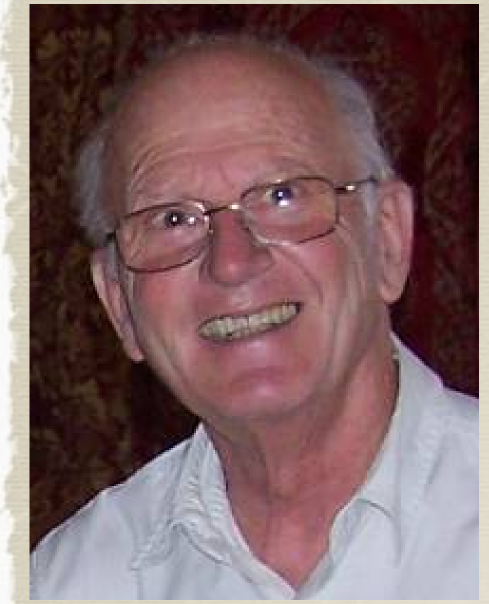
J.-M. LÉVY-LEBLOND
Physique Théorique - Paris (***)

(ricevuto il 6 Novembre 1972)

Summary. — Consistent nonrelativistic electromagnetic theories are investigated by stressing the requirements of Galilean relativity. It is shown that Maxwell's equations admit two possible nonrelativistic limits, accounting respectively for electric and magnetic effects. A Galilean theory is then built which combines these two theories and can embody a large class of experimental facts. As a result, several so-called « relativistic » effects are shown to necessitate a re-appraisal, or at least, a more careful discussion. It is finally shown precisely how the old-fashioned formulation of the electromagnetic theory in terms of field strengths and field excitations clashes with Galilean relativity in its constitutive equations only, leading to the idea of a privileged frame of reference (the ether) or to Einsteinian relativity!

M. Le Bellac and J.-M. Lévy-Leblond, Galilean Electromagnetism, Il Nuovo Cimento, 14B, 217 (1973)

Galilean Limits Of Electromagnetism



$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \frac{(1-\gamma)(\mathbf{v}.\mathbf{E})\mathbf{v}}{\mathbf{v}^2}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \left(1/c_L^2\right)\mathbf{v} \times \mathbf{E}\right) + \frac{(1-\gamma)(\mathbf{v}.\mathbf{B})\mathbf{v}}{\mathbf{v}^2}$$

$$\gamma = \left(1 - \mathbf{v}^2/c^2\right)^{-1/2}$$

$$\gamma \to 1$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$\mathbf{B}' = \mathbf{B} - \left(1/c_L^2\right)\mathbf{v} \times \mathbf{E}$$

$$E >> c_L B$$

$$\mathbf{E}' = \mathbf{E}$$

$$\mathbf{B}' = \mathbf{B} - \left(1/c_L^2\right)\mathbf{v} \times \mathbf{E}$$

Electric Limit

$$\nabla.\mathbf{E} = \rho/\varepsilon_0, \quad \nabla.\mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = 0, \quad \nabla \times \mathbf{B} = \left(1/c^2\right)\partial\mathbf{E}/\partial t + \mu_0\mathbf{J}.$$

$$\rho'(\mathbf{x}', t') = \rho(\mathbf{x}, t)$$

$$\mathbf{J}'(\mathbf{x}', t') = \mathbf{J}(\mathbf{x}, t) - \mathbf{v}\rho(\mathbf{x}, t).$$

$$E << c_L B$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

$$\mathbf{B}' = \mathbf{B}$$

Magnetic Limit

$$\nabla.\mathbf{E} = \rho/\varepsilon_0, \quad \nabla.\mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t, \quad \nabla \times \mathbf{B} = \mu_0\mathbf{J}$$

$$\rho'(\mathbf{x}', t') = \rho(\mathbf{x}, t) - \left(1/c^2\right)\mathbf{v}.\mathbf{J}(\mathbf{x}, t)$$

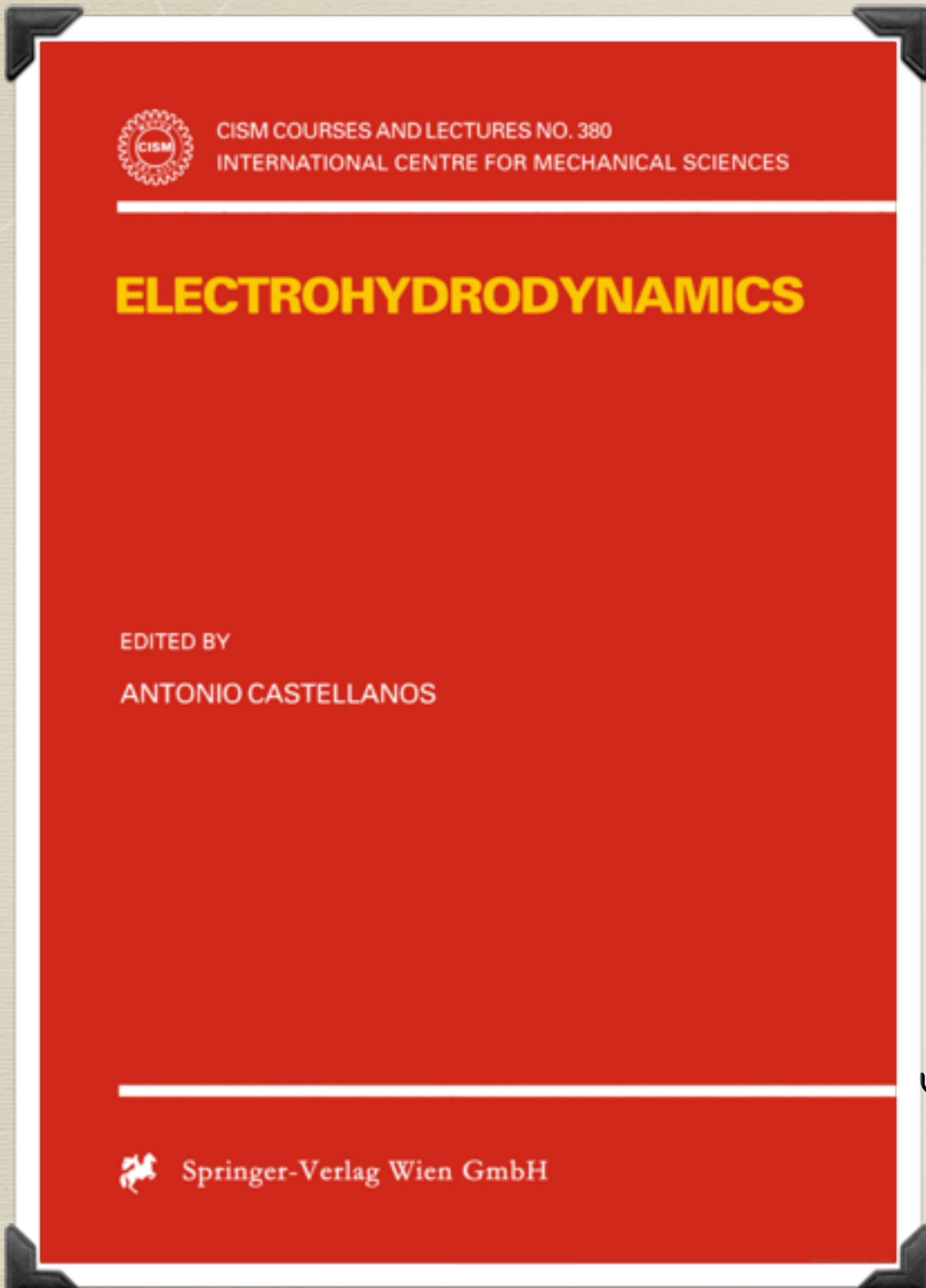
$$\mathbf{J}'(\mathbf{x}', t') = \mathbf{J}(\mathbf{x}, t)\,. \qquad \nabla.\mathbf{J} = 0$$

M. Le Bellac and J.-M. Lévy-Leblond, Galilean Electromagnetism, Il Nuovo Cimento, 14B, 217 (1973)

Personal Reminiscences

I was trained both in MHD and **EHD in Grenoble**, France during the years 1997-2000
both in the engineer schools of Physics/Hydraulics and in the University.
In MHD, I followed the lectures of Professor René Moreau who gave to me his lecture notes
from the Summer School in Les Houches on EHD in the early 70's !
In EHD, I followed the lectures of Nelly Bonifaci, a colleague of **Pierre Atten**.
I was absolutely astonished to see that two distinct sets of Maxwell's equations
were used separately in EHD and MHD within the approximation of the so-called quasi-statics.
In France, the quasi-statics approximation was usually (but wrongly) thought as a synonym of only MQS.
I did my Phd Thesis in ESPCI during the years 2000-2003 with **José-Eduardo Wesfreid**,
a friend of Antonio on the Physics of **Granular Matter**.
I was also working with Etienne Guyon on the analogy used by Maxwell between fluids mechanics (Galilean covariant) and
electromagnetism (Lorentz covariant) : how can there be an analogy whereas the kinematics are different ?
I read with very much interest the lecture notes of Antonio on EHD and the paper by Le Bellac and Lévy-Leblond
on Galilean Electromagnetism during this period : Maxwell's equations were also compatible with Galilean Physics !
In ESPCI, I met and discussed with Antonio who came for a seminar
on granular matter with electrostatics effects.
We barely discussed of **Galilean Electromagnetism** but latter I sent him my papers on the subject
where he was duly quoted as a source of inspiration.
Antonio seemed to have been unaware of the work of Le Bellac and Lévy-Leblond
on Galilean Electromagnetism that he never quoted as far as I know.
He never discussed the Galilean limits of neither the scalar and vector potentials nor the gauge conditions.

Antonio Castellanos as a Reader of James Melcher

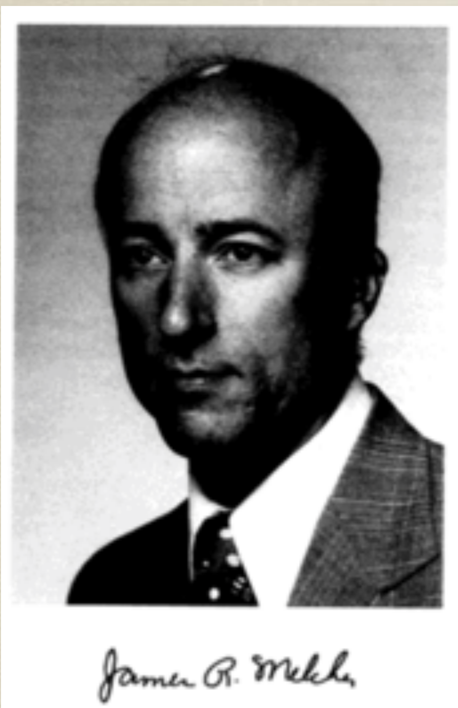


This book may be considered as complementary to the excellent treatment of EHD made in the classical book "Continuum Electromechanics" by Melcher, in that care has been taken to avoid overlapping of the subjects. In case a topic is treated in both texts, the results presented in the book by Melcher serve as an introduction to the more advanced treatment presented in this book.

James R. Melcher, Continuum Electromechanics. Cambridge, MA: MIT Press (1981).

A. Castellanos. Basic concepts and equations in electrohydrodynamics (Chapters 1-4). In: A. Castellanos (editor), Electrohydrodynamics, Springer Verlag, Wien – New York (1998).

The EQS and MQS Systems of Maxwell Equations



To find the first order contributions, these equations with $n=1$ are solved with the zero order solutions making up the right-hand sides of the equations playing the role of known driving functions. Boundary conditions are satisfied by the lowest order fields. Thus higher order fields satisfy homogeneous boundary conditions.

Once the first order solutions are known, the process can be repeated with these forming the "drives" for the $n=2$ equations.

In the absence of loss effects, there are no characteristic times to distinguish MQS and EQS systems. In that limit, which set of normalizations is used is a matter of convenience. If a situation represented by the left-hand set actually has an EQS limit, the zero order laws become the quasistatic laws. But, if these expressions are applied to a situation that is actually MQS, then first-order terms must be calculated to find the quasistatic fields. If more than the one characteristic time τ_{em} is involved, as is the case with finite τ_e and τ_m, then the ordering of rate parameters can contribute to the convergence of the expansion.

In practice, a formal derivation of the quasistatic laws is seldom used. Rather, intuition and experience along with comparison of critical time constants to relevant dynamical times is used to identify one of the two sets of zero order expressions as appropriate. But, the use of normalizations to identify critical parameters, and the notion that characteristic times can be used to unscramble dynamical processes, will be used extensively in the chapters to follow.

Within the framework of quasistatic electrodynamics, the unnormalized forms of Eqs. 13-17 comprise the "exact" field laws. These equations are reordered to reflect their relative importance:

Electroquasistatic (EQS)

$$\nabla\cdot\epsilon_o\vec{E} = -\nabla\cdot\vec{P} + \rho_f$$

$$\nabla\times\vec{E} = 0$$

$$\nabla\cdot\vec{J}_f + \frac{\partial\rho_f}{\partial t} = 0$$

$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\epsilon_o\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t} + \nabla\times(\vec{P}\times\vec{v})$$

$$\nabla\cdot\mu_o\vec{H} = -\nabla\,\mu_o\vec{M}$$

Magnetoquasistatic (MQS)

$$\nabla\times\vec{H} = \vec{J}_f \qquad (23)$$

$$\nabla\cdot\mu_o\vec{H} = -\nabla\cdot\mu_o\vec{M} \qquad (24)$$

$$\nabla\times\vec{E} = -\frac{\partial\mu_o\vec{H}}{\partial t} - \frac{\partial\mu_o\vec{M}}{\partial t} - \mu_o\nabla\times(\vec{M}\times\vec{v}) \qquad (25)$$

$$\nabla\cdot\vec{J}_f = 0 \qquad (26)$$

$$\nabla\cdot\epsilon_o\vec{E} = -\nabla\cdot\vec{P} + \rho_f \qquad (27)$$

The conduction current $\vec{J}_f$ has been reintroduced to reflect the wider range of validity of these equations than might be inferred from Eq. 1. With different conduction models will come different characteristic times, exemplified in the discussions of this section by τ_e and τ_m. Matters are more complicated if fields and media interact electromechanically. Then, $\vec{v}$ is determined to some extent at least by the fields themselves and must be treated on a par with the field variables. The result can be still more characteristic times.

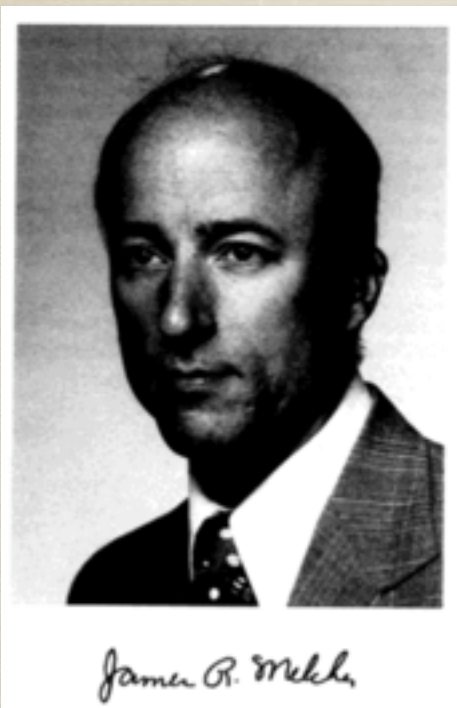
The ordering of the quasistatic equations emphasizes the instantaneous relation between the respective dominant sources and fields. Given the charge and polarization densities in the EQS system, or given the current and magnetization densities in the MQS system, the dominant fields are known and are functions only of the sources at the given instant in time.

The dynamics enter in the EQS system with conservation of charge, and in the MQS system with Faraday's law of induction. Equations 26a and 27a are only needed if an after-the-fact determination of $\vec{H}$ is to be made. An example where such a rare interest in $\vec{H}$ exists is in the small magnetic field induced by electric fields and currents within the human body. The distribution of internal fields and hence currents is determined by the first three EQS equations. Given $\vec{E}$, $\vec{P}$, and $\vec{J}_f$, the remaining two expressions determine $\vec{H}$. In the MQS system, Eq. 27b can be regarded as an expression for the after-the-fact evaluation of ρ_f, which is not usually of interest in such systems.

What makes the subject of quasistatics difficult to treat in a general way, even for a system of fixed ohmic conductivity, is the dependence of the appropriate model on considerations not conveniently represented in the differential laws. For example, a pair of perfectly conducting plates, shorted on one pair of edges and driven by a sinusoidal source at the opposite pair, will be MQS at low frequencies. The same pair of plates, open-circuited rather than shorted, will be electroquasistatic at low frequencies. The difference is in the boundary conditions.

Geometry and the inhomogeneity of the medium (insulators, perfect conductors and semiconductors) are also essential to determining the appropriate approximation. Most systems require more than one

http://www.rle.mit.edu/cehv/documents/ContinuumElectromechanics-Melcher.pdf



The most convenient normalization of the fields depends on the specific system. Where electromechanical coupling is significant, these can usually be categorized as "electric-field dominated" and "magnetic-field dominated." Anticipating this fact, two normalizations are now developed "in parallel," the first taking $\mathcal{E}$ as a characteristic electric field and the second taking $\mathcal{H}$ as a characteristic magnetic field:

$$\vec{E} = \mathcal{E}\underline{\vec{E}},\ \vec{P} = \epsilon_o\mathcal{E}\underline{\vec{P}},\ \vec{v} = (\ell/\tau)\underline{\vec{v}},\ \vec{J}_v = \frac{\epsilon_o\mathcal{E}}{\tau}\underline{\vec{J}_v},\quad \rho_f = \frac{\mathcal{E}\epsilon_o}{\ell}\underline{\rho_f},\ \vec{H} = \frac{\epsilon_o\ell\mathcal{E}}{\tau}\underline{\vec{H}},\ \vec{M} = \frac{\epsilon_o\ell\mathcal{E}}{\tau}\underline{\vec{M}}$$

$$\vec{H} = \mathcal{H}\underline{\vec{H}},\ \vec{M} = \mathcal{H}\underline{\vec{M}},\ \vec{v} = (\ell/\tau)\underline{\vec{v}},\ \vec{J}_v = \frac{\mathcal{H}}{\ell}\underline{\vec{J}_v},\quad \vec{E} = \frac{\mu_o\ell\mathcal{H}}{\tau}\underline{\vec{E}},\ \rho_f = \frac{\epsilon_o\mu_o\mathcal{H}}{\tau}\underline{\rho_f},\ \vec{P} = \frac{\epsilon_o\mu_o\ell\mathcal{H}}{\tau}\underline{\vec{P}} \qquad (4)$$

It might be appropriate with this step to recognize that the material motion introduces a characteristic (transport) time other than τ. For simplicity, Eq. 4 takes the material velocity as being of the order of ℓ/τ.

The normalization used is arbitrary. The same quasistatic laws will be deduced regardless of the starting point, but the normalization will determine whether these laws are "zero-order" or higher order in a sense to now be defined.

The normalizations of Eq. 4 introduced into Eqs. 2.2.1-5 result in

$$\nabla\cdot\vec{E} = -\nabla\cdot\vec{P} + \rho_f \qquad\Big|\qquad \nabla\cdot\vec{E} = -\nabla\cdot\vec{P} + \rho_f \qquad (\underline{5})$$

$$\nabla\cdot\vec{H} = -\nabla\cdot\vec{M} \qquad\Big|\qquad \nabla\cdot\vec{H} = -\nabla\cdot\vec{M} \qquad (\underline{6})$$

$$\nabla\times\vec{H} = \frac{\tau}{\tau_e}\sigma\vec{E} + \vec{J}_v + \frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{P}}{\partial t} + \nabla\times(\vec{P}\times\vec{v}) \qquad\Big|\qquad \nabla\times\vec{H} = \frac{\tau_m}{\tau}\sigma\vec{E} + \vec{J}_v + \beta\left[\frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \nabla\times(\vec{P}\times\vec{v})\right] \qquad (\underline{7})$$

$$\nabla\times E = -\beta\left[\frac{\partial\vec{H}}{\partial t} + \frac{\partial\vec{M}}{\partial t} + \nabla\times(\vec{M}\times\vec{v})\right] \qquad\Big|\qquad \nabla\times\vec{E} = -\frac{\partial\vec{H}}{\partial t} - \frac{\partial\vec{M}}{\partial t} - \nabla\times(\vec{M}\times\vec{v}) \qquad (\underline{8})$$

$$\nabla\cdot\sigma\vec{E} + \frac{\tau_e}{\tau}\left[\nabla\cdot\vec{J}_v + \frac{\partial\rho_f}{\partial t}\right] = 0 \qquad\Big|\qquad \nabla\cdot\sigma\vec{E} + \frac{\tau}{\tau_m}\nabla\cdot\vec{J}_v + \beta\frac{\tau}{\tau_m}\frac{\partial\rho_f}{\partial t} = 0 \qquad (\underline{9})$$

where underbars on equation numbers are used to indicate that the equations are normalized and

$$\tau_m \equiv \mu_o\sigma_o\ell^2,\ \tau_e \equiv \epsilon_o/\sigma_o$$

and

$$\beta = \left(\frac{\tau_{em}}{\tau}\right)^2;\ \tau_{em} \equiv \sqrt{\mu_o\epsilon_o}\,\ell = \ell/c \qquad (10)$$

In Chap. 6, τ_m will be identified as the magnetic diffusion time, while in Chap. 5 the role of the charge-relaxation time τ_e is developed. The time required for an electromagnetic plane wave to propagate the distance ℓ at the velocity c is τ_{em}. Given that there is just one characteristic length, there are actually only two characteristic times, because as can be seen from Eq. 10

$$\sqrt{\tau_m\tau_e} = \tau_{em} \qquad (11)$$

Unless τ_e and τ_m, and hence τ_{em}, are all of the same order, there are only two possibilities for the relative magnitudes of these times, as summarized in Fig. 2.3.1.

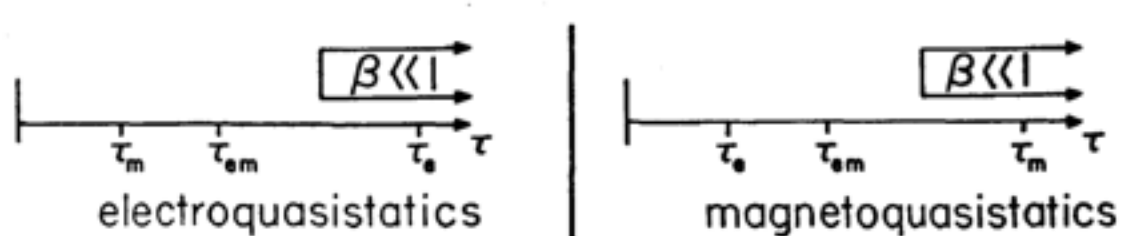


Fig. 2.3.1. Possible relations between physical time constants on a time scale τ which typifies the dynamics of interest.

The Normalisation Procedure to Derive EQS and MQS

http://www.rle.mit.edu/cehv/documents/ContinuumElectromechanics-Melcher.pdf

The Energetics Approach to EQS and MQS à la Castellanos

To choose a scale to nondimensionalize the EM fields we need to compare the relative magnitude of the electric to the magnetic field. For radiation fields, such as plane waves, it is well known that $E = cB$ with E the magnitude of the electric field, c the velocity of light, and B the magnitude of the magnetic field. Also, it is well known from relativistic electromagnetism that, $E^2 - c^2B^2$, is a Lorentz invariant that may be positive, zero or negative. Alternatively, for those people not familiar with relativity, it may be easier to think in terms of energies. We have

$$\frac{\varepsilon E^2/2}{B^2/2\mu} = \frac{E^2}{c^2B^2}. \tag{1.13}$$

We recover again the three previous cases, which now correspond to an electrically dominated system, a radiation field, and a magnetically dominated system, when this ratio is greater, equal, or lesser than one. We neglect the particular case in which we may have an electrostatic field and a magnetostatic field, both decoupled from each other. In the first case, denoted as Electroquasistatics, and in the third case, named Magnetoquasistatics, it is possible to simplify the Maxwell's equations. In both cases all the intrincacies due to electromagnetic wave phenomena and radiation may be disregarded, and as a consequence a great degree of simplification is achieved. Moreover, in the two latter cases it is possible to take the limit, $c \to \infty$, in a consistent way, thus obtaining two different Galilean limits to the electromagnetic field, with corresponding field transformations specific to each case.

A. Castellanos. Basic concepts and equations in electrohydrodynamics (Chapters 1-4). In: A. Castellanos (editor), Electrohydrodynamics, Springer Verlag, Wien – New York (1998).

1.3 Electroquasistatics in moving fluids

In this section we elaborate the case of fluids subjected to electroquasistatic fields a little further. Consider first that the fluid is subjected to a time varying electric field of frequency ω. As discussed in the previous section, a necessary condition to reduce Maxwell's equations to the set of electroquasistatics equations, is that $\tau_{em} \ll \tau$ is true where τ is any characteristic time associated with electrical or mechanical processes taking place in the fluid. In particular, for a time varying electric field we may take as τ the period, T, of the electric field, and therefore we must have

$$\frac{l}{cT} \sim \frac{l\omega}{c} \ll 1, \tag{1.58}$$

The condition $\omega \ll (c/l)$ ensures that the magnetic field generated by the displacement current, which is of order $B_0 \simeq \mu\varepsilon l\omega E_0$, satisfies automatically the condition $cB_0 \ll E_0$.

The second condition that must be met by the physical system in order to be quasielectrostatic, is that the magnetic field due to the current density in the system, regardless of its origin, i. e. injection, dissociation or current due to particles, must also satisfy the condition $cB_0 \ll E_0$. Since from the fourth Maxwell's equation the scale for B_0 due to the current density is $B_0 = \mu l J_0$, we have

$$\frac{cB_0}{E_0} = \frac{cl\mu J_0}{E_0}, \tag{1.59}$$

and consequently

$$J_0 \ll E_0\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{l}. \tag{1.60}$$

For typical systems in the laboratory $l \sim 10^{-2}$m, $\mu \simeq \mu_0 \simeq 10^{-6}\mathrm{Hm}^{-1}$ and $\varepsilon \sim 10^{-11}\mathrm{Fm}^{-1}$, and we must have $J_0 \ll 3E_0$. In insulating liquids this condition is amply satisfied in practical situations as J_0 is in the range of microamperes per square centimetre, and E_0 is several kV per centimeter.

To conclude, in Electrohydrodynamics, Maxwell equations reduce to

$$\nabla \cdot \mathbf{D} = q, \qquad \nabla \times \mathbf{E} = 0, \qquad \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{1.61}$$

with the magnetic field determined by the equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.62}$$

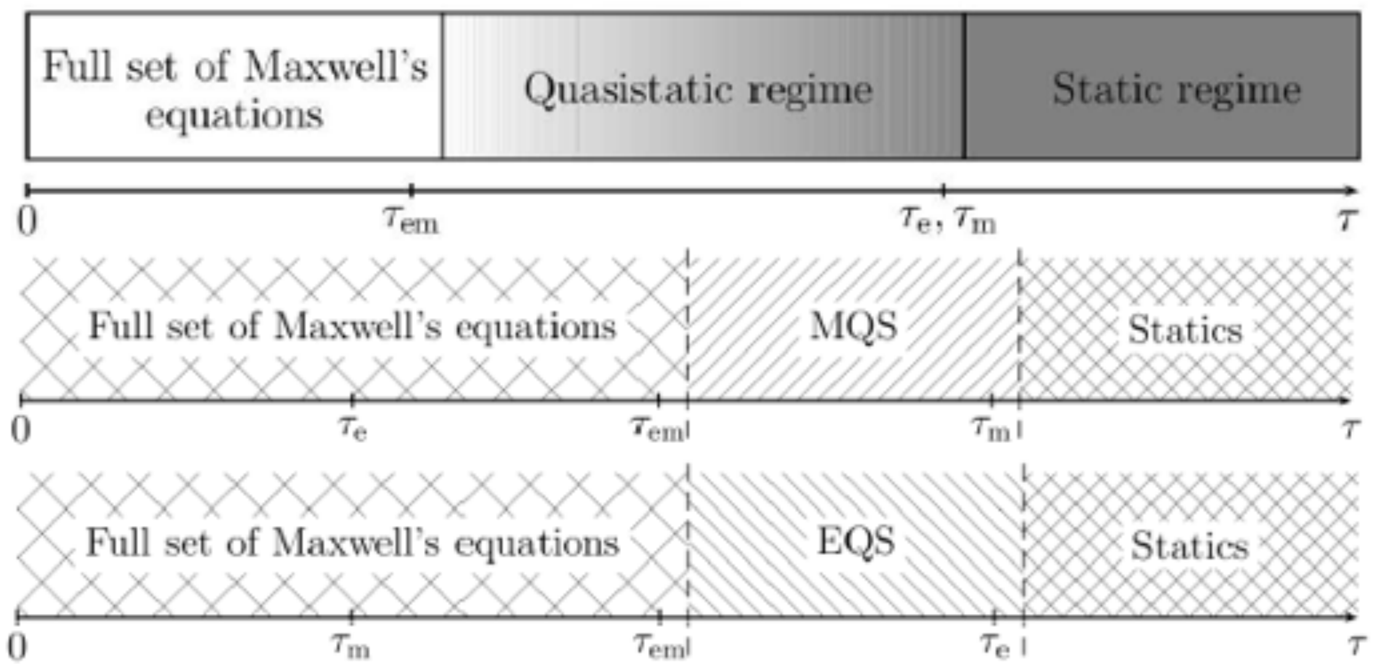
A. Castellanos. Basic concepts and equations in electrohydrodynamics (Chapters 1-4). In: A. Castellanos (editor), Electrohydrodynamics, Springer Verlag, Wien – New York (1998).

Characteristic times

$\tau_{em} = \ell/c$ the light transit time in the medium (here, $\tau_{em} \ll \tau$)

$\tau_e = \epsilon/\sigma$ the charge relaxation time

$\tau_m = \mu\sigma\ell^2$ the magnetic diffusion time, $\tau_{em} = \sqrt{\tau_e \tau_m}$



	Corn oil	Water	Copper
ε	$3.1\varepsilon_0$	$81\varepsilon_0$	$\approx \varepsilon_0$
σ [S/m]	5×10^{-11}	0.2	5.7×10^{7}
ℓ^* [m]	10^{8}	0.12	4.7×10^{-11}
$\log\left(\frac{\ell}{\ell^*}\right)$	-8	1	11
τ_{em}[s]	5.8×10^{-9}	3×10^{-8}	3×10^{-9}
Full set	$\tau < \tau_{\mathrm{em}}$	$\tau < \tau_{\mathrm{em}}$	$\tau < \tau_{\mathrm{em}}$
EQS	$\tau_{\mathrm{em}} < \tau < 5.8 \times 10^{-1}$ s	–	–
MQS	–	$\tau_{\mathrm{em}} < \tau < 3 \times 10^{-7}$ s	$\tau_{\mathrm{em}} < \tau < 3 \times 10^{2}$ s
Statics	$\tau > 5.8 \times 10^{-1}$ s	$\tau > 3 \times 10^{-7}$ s	$\tau > 3 \times 10^{2}$ s

Orders of magnitude

$E = e\mathcal{E}$, with e a reference quantity and $\mathcal{E}$ a non-dimensional vector

τ, a characteristic time constant ℓ, a characteristic spatial dimension,

$\ell^* = \frac{1}{\sigma}\sqrt{\frac{\epsilon}{\mu}} = \frac{1}{\sigma\eta}$ is the constitutive length (η medium impedance)

$v = \frac{\ell}{\tau}$ velocity of the phenomenon in the medium or of the medium

$c = \frac{1}{\sqrt{\mu\epsilon}}$ velocity of light in the medium

$\partial_x E$ approximated by e/ℓ, $\partial_t E$ approximated by e/τ

In the empty space, two velocities and scalings appearing :

$$\nabla \times E = -\partial_t B \quad \rightarrow \quad \frac{e}{\ell} \sim \frac{b}{\tau} \quad \rightarrow \quad e \sim vb \quad (v = \tfrac{\ell}{\tau})$$

$$\nabla \times H = \partial_t D \rightarrow \quad \frac{b}{\mu_0 \ell} \sim \frac{\epsilon_0 e}{\tau} \quad \rightarrow \quad b \sim \tfrac{v}{c^2} e \quad (c = \tfrac{1}{\sqrt{\mu_0\epsilon_0}}).$$

$v \approx c$ yields $\begin{cases} e = cb \\ b = \dfrac{e}{c} \end{cases}$, and $v \ll c$ yields $\begin{cases} e = vb \\ b = \frac{v}{c^2} e \end{cases}$ thus $v = c$, wrong !

This means that for $v \ll c$, the two scalings are not simultaneously valid thus Faraday and Ampère laws cannot be coupled

To choose the scaling, look at the field sources (charges or currents) !

Galilean limits of Maxwell equations

For fields

If $J \gg \rho c$ then $e \ll cb$, thus MQS (the dielectric effect of charges is negligible)

Galilean magnetic limit, known as MagnetoQuasiStatics (MQS),

$$\text{where } v \ll c \text{ and } e \sim vb: \quad \begin{cases} \nabla \times E = -\partial_t B, \\ \nabla \cdot B = 0, \\ \nabla \times H \simeq J, \\ \nabla \cdot J = 0 \end{cases}$$

If $J \ll \rho c$, then $e \gg cb$, thus EQS (the conducting effect is negligible)

Galilean electric limit, known as ElectroQuasiStatics (EQS),

$$\text{where } v \ll c \text{ and } b \sim \tfrac{v}{c^2}e: \quad \begin{cases} \nabla \times E \simeq 0, \\ \nabla \cdot B = 0, \\ \nabla \times H = J + \partial_t D, \\ \nabla \cdot D = \rho \end{cases}$$

Galilean stationary limit, known as QuasiStationaryConduction (QSC), where fields exhibit no time-dependence.

For potentials

$A = a\mathcal{A}$ (magnetic vector potential), $V = \mathtt{V}\mathcal{V}$ (electric scalar potential)

$B = \nabla \times A$ yields $b \sim \frac{a}{\ell}$ (always true)

$E = -\partial_t A - \nabla V$ yields $e \sim \frac{a}{\tau} + \frac{\mathtt{V}}{\ell}$ thus

• if $c\,a \ll \mathtt{V}$ ($a \approx \frac{v}{c^2}\mathtt{V}$) then $(v \ll c)$ $v\,a \ll \mathtt{V}$, $\frac{\ell}{\tau}a \ll \mathtt{V}$, $\frac{a}{\tau} \ll \frac{\mathtt{V}}{\ell}$ so $e \sim \frac{\mathtt{V}}{\ell}$ (EQS).

• if $c\,a \gg \mathtt{V}$ ($\mathtt{V} \approx v\,a$) then $a \gg \mathtt{V}$ is compensated by $v \ll c$ so $\frac{a}{\tau} \sim \frac{\mathtt{V}}{\ell}$ (MQS)

For gauge conditions

In the empty space :

• if $c\,a \ll \mathtt{V}$ then $\nabla \cdot A + \frac{1}{c^2}\partial_t V = 0$ (Lorenz)

• if $c\,a \gg \mathtt{V}$ then $\nabla \cdot A = 0$ (Coulomb)

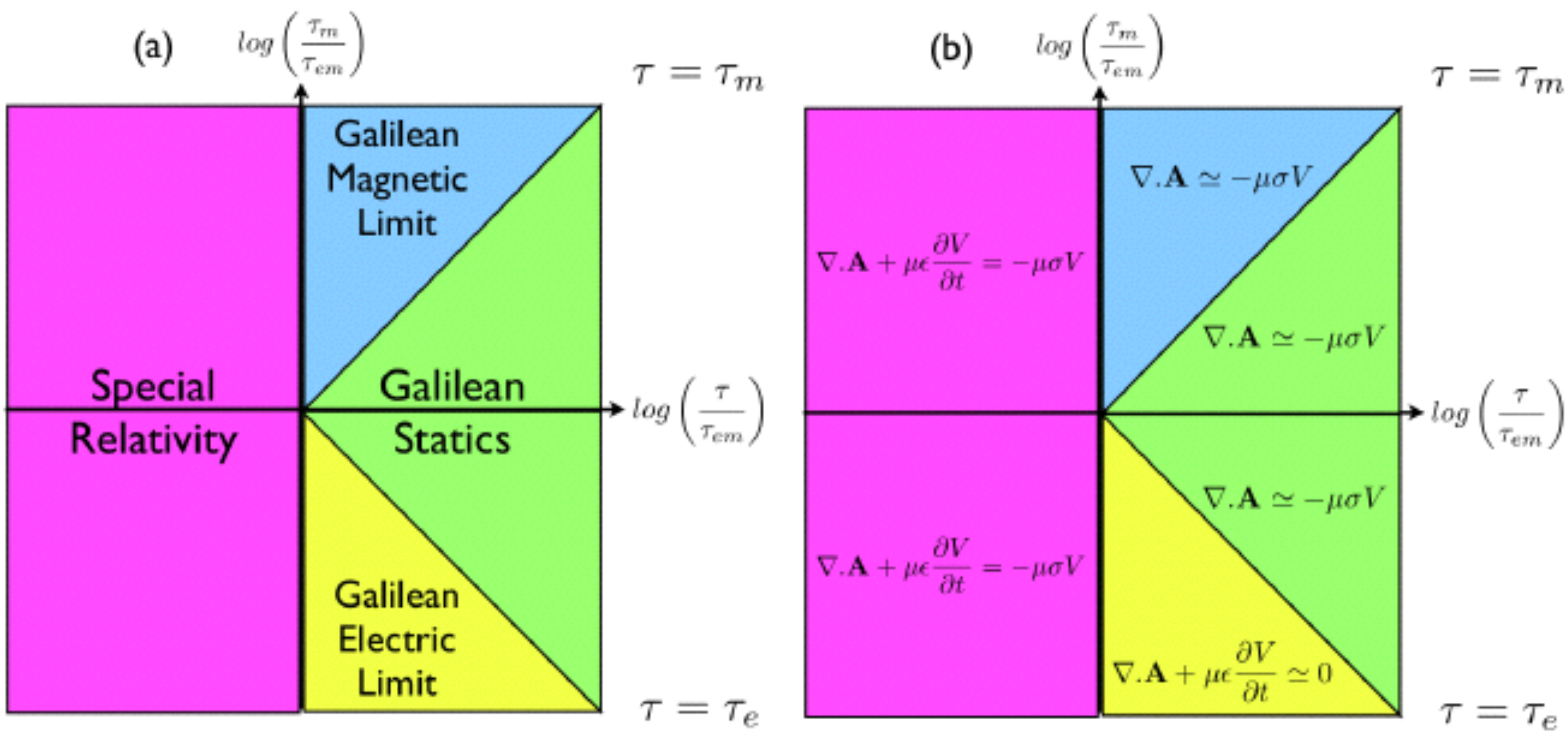
In the medium with a finite σ, $\nabla \cdot A + \mu\epsilon\partial_t V = -\mu\sigma V$ (Stratton)

From Stratton gauge condition with orders of magnitude

$$\nabla' \cdot \mathcal{A} + \frac{\frac{\ell}{\tau}}{c}\frac{\mathtt{V}}{ca}\partial_{t'}\mathcal{V} = -\frac{\ell}{\ell^*}\frac{\mathtt{V}}{ca}\mathcal{V}, \qquad I + II = III$$

Dimensionless ratios

$$\frac{II}{I} \sim \frac{v}{c}\frac{\mathtt{V}}{ca} = \frac{\tau_{em}}{\tau}\frac{\mathtt{V}}{ca}, \qquad \frac{III}{I} \sim \frac{\ell}{\ell^*}\frac{\mathtt{V}}{ca} = \frac{\tau_m}{\tau_{em}}\frac{\mathtt{V}}{ca} = \frac{\tau_{em}}{\tau_e}\frac{\mathtt{V}}{ca}, \qquad \frac{III}{II} \sim \frac{\tau}{\tau_e}$$



$$x = log(\frac{\tau}{\tau_{em}}), \quad y = log(\frac{\ell}{\ell^*}) = log(\frac{\tau_m}{\tau_{em}})$$

James R. Melcher

proof. The approach is as follows. First, the postulate is made that the quasistatic equations take th same form in the primed and unprimed inertial reference frames. But, in writing the laws in the primed frame, the spatial and temporal derivatives must be taken with respect to the coordinates of that reference frame, and the dependent field variables are then fields defined in that reference frame. In general, these must be designated by primes, since their relation to the variables in the unprimed frame is not known.

For the purpose of writing the primed equations of electrodynamics in terms of the unprimed coordinates, recognize that

$$\nabla' \rightarrow \nabla$$

$$\frac{\partial \vec{A}}{\partial t'} \rightarrow \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{A} \equiv \frac{\partial \vec{A}}{\partial t} + \vec{u}\nabla\cdot\vec{A} - \nabla \times (\vec{u} \times \vec{A}) \tag{2}$$

$$\frac{\partial \psi}{\partial t'} \rightarrow \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\psi \equiv \frac{\partial \psi}{\partial t} + \nabla\cdot\vec{u}\psi$$

The left relations follow by using the chain rule of differentiation and the transformation of Eq. 1. That the spatial derivatives taken with respect to one frame must be the same as those with respect to the other frame physically means that a single "snapshot" of the physical process would be all required to evaluate the spatial derivatives in either frame. There would be no way of telling which frame was the one from which the snapshot was taken. By contrast, the time rate of change for an observer in the primed frame is, by definition, taken with the primed spatial coordinates held fixed. In terms of the fixed frame coordinates, this is the convective derivative defined with Eqs. 2.4.5 and 2.4.6. However, $\vec{v}$ in these equations is in general a function of space and time. In the context of this section it is specialized to the constant $\vec{u}$. Thus, in rewriting the convective derivatives of Eq. 2 the constancy of $\vec{u}$ and a vector identity (Eq. 16, Appendix B) have been used.

So far, what has been said in this section is a matter of coordinates. Now, a physically motivate postulate is made concerning the electromagnetic laws. Imagine one electromagnetic experiment that is to be described from the two different reference frames. The postulate is that provided each of these frames is inertial, the governing laws must take the same form. Thus, Eqs. 23-27 apply with $[\nabla \rightarrow \nabla'$, $\partial(\ )/\partial t \rightarrow \partial(\ )/\partial t']$ and all dependent variables primed. By way of comparing these laws to those expressed in the fixed-frame, Eqs. 2 are used to rewrite these expressions in terms of the unprimed independent variables. Also, the moving-frame material velocity is rewritten in terms of the unprimed frame velocity using the relation

$$\vec{v}' = \vec{v} - \vec{u} \tag{3}$$

Thus, the laws originally expressed in the primed frame of reference become

$$\nabla\cdot\epsilon_o\vec{E}' = -\nabla\cdot\vec{P}' + \rho_f' \qquad\qquad \nabla \times \vec{H}' = \vec{J}_f' \tag{4}$$

$$\nabla \times \vec{E}' = 0 \qquad\qquad \nabla\cdot\mu_o\vec{H}' = -\nabla\cdot\mu_o\vec{M}' \tag{5}$$

$$\nabla\cdot(\vec{J}_f' + \vec{u}\rho_f') + \frac{\partial \rho_f'}{\partial t} = 0 \qquad\qquad \nabla\times(\vec{E}' - \vec{u} \times \mu_o\vec{H}') = -\frac{\partial \mu_o\vec{H}'}{\partial t} - \frac{\partial \mu_o\vec{M}'}{\partial t} - \mu_o\nabla \times (\vec{M}' \times \vec{v}) \tag{6}$$

$$\nabla \times (\vec{H}' + \vec{u} \times \epsilon_o\vec{E}') = (\vec{J}_f' + \vec{u}\rho_f') + \frac{\partial \epsilon_o\vec{E}'}{\partial t} + \frac{\partial \vec{P}'}{\partial t} + \nabla \times (\vec{P}' \times \vec{v}) \qquad\qquad \nabla\cdot\vec{J}_f' = 0 \tag{7}$$

$$\nabla\cdot\mu_o\vec{H}' = -\nabla\cdot\mu_o\vec{M}' \qquad\qquad \nabla\cdot\epsilon_o\vec{E}' = -\nabla\cdot\vec{P}' + \rho_f' \tag{8}$$

In writing Eq. 7a, Eq. 4a is used. Similarly, Eq. 5b is used to write Eq. 6b. For the one experiment under consideration, these equations will predict the same behavior as the fixed frame laws, Eqs. 2.3.23-27, if the identification is made:

The Galilean Procedure to Derive EQS and MQS

http://www.rle.mit.edu/cehv/documents/ContinuumElectromechanics-Melcher.pdf

For fields

Magnetic Limit	Electric Limit
$\rho = \rho' + v \cdot J'/c^2$	$\rho = \rho'$
$J = J'$	$J = J' - \rho' v$
$B = B'$	$B = B' + v \times E'/c^2$
$E = E' - v \times B'$	$E_e = E'$
$H = H'$	$H = H' + v \times D'$
$D = D' - v \times H'/c^2$	$D = D'$
$M = M'$	$M = M' - v \times P'$
$P = P' + v \times M'/c^2$	$P = P'$

For potentials

Magnetic Limit	Electric Limit
$A' = A$	$A' = A - v\frac{V}{c^2}$
$V' = V - v \cdot A$	$V' = V$

For constitutive relations

Magnetic Limit	Electric Limit
$D' \simeq \epsilon E + (\epsilon - \frac{1}{\mu c^2}) v \times B$	$D' \simeq \epsilon E$
$H' \simeq \frac{B}{\mu}$	$H' \simeq \frac{B}{\mu} + (\epsilon - \frac{1}{\mu c^2}) \frac{v \times E}{c^2}$

Muchas Gratias Antonio for helping me to bridge the gap between the separate approches to Galilean Electromagnetism of the engineers à la Melcher and of the physicists à la Le Bellac and Lévy-Leblond !

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THE EUROPEAN PHYSICAL JOURNAL PLUS

Review

Forty years of Galilean Electromagnetism (1973–2013)

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Abstract. We review Galilean Electromagnetism since the 1973 seminal paper of Jean-Marc Lévy-Leblond and Michel Le Bellac and we explain for the first time all the historical experiments of Rowland, Vasilescu Karpen, Roentgen, Eichenwald, Wilson, Wilson and Wilson, which were previously interpreted in a Special Relativistic framework by showing the uselessness of the latter for setups involving slow motions of a part of the apparatus. Galilean Electromagnetism is not an alternative to Special Relavity but is precisely its low-velocity limit in Classical Electromagnetism.

EUROPHYSICS LETTERS 1 July 2005

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Lorenz or Coulomb in Galilean electromagnetism?

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PACS. 03.50.De – Classical electromagnetism, Maxwell equations.
PACS. 41.20.-q – Applied classical electromagnetism.
PACS. 47.65.+a – Magnetohydrodynamics and electrohydrodynamics.

Abstract. – Galilean electromagnetism was discovered thirty years ago by Lévy-Leblond and Le Bellac. However, these authors only explored the consequences for the fields and not for the potentials. Following De Montigny *et al.*, we show that the Coulomb gauge condition is the magnetic limit of the Lorenz gauge condition whereas the Lorenz gauge condition applies in the electric limit of Lévy-Leblond and Le Bellac. Contrary to De Montigny *et al.*, who used Galilean tensor calculus, we use orders of magnitude based on physical motivations in our derivation.

On some applications of Galilean electrodynamics of moving bodies

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We discuss the seminal article by Le Bellac and Lévy-Leblond in which they identified two Galilean limits (called "electric" and "magnetic" limits) of electromagnetism and their implications. Recent work has shed new light on the choice of gauge conditions in classical electromagnetism. We show that the recourse to potentials is compelling in order to demonstrate the existence of both (electric and magnetic) limits. We revisit some nonrelativistic systems and related experiments, in the light of these limits, in quantum mechanics, superconductivity, and the electrodynamics of continuous media. Much of the current technology where waves are not taken into account can be described in a coherent fashion by the two limits of Galilean electromagnetism instead of an inconsistent mixture of these limits.

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On the electrodynamics of Minkowski at low velocities

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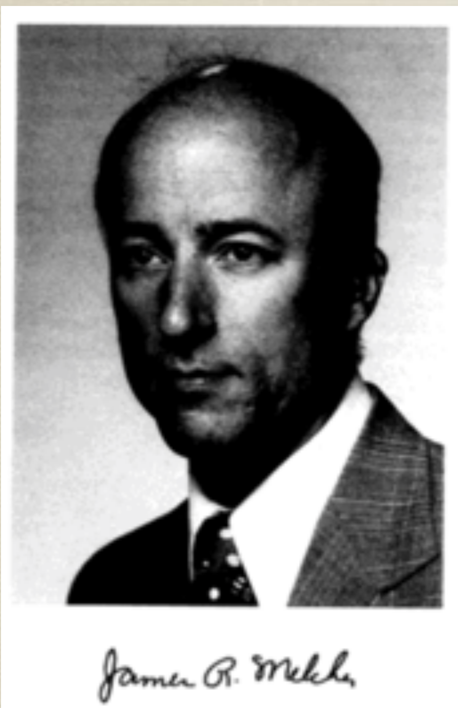
PACS 03.50.De – Classical electromagnetism, Maxwell equations

Abstract – The Galilean constitutive equations for the electrodynamics of moving media are derived for the first time. They explain all the historic and modern experiments which were interpreted so far in a relativistic framework assuming the constant light celerity principle. Here, we show the latter to be sufficient but not necessary.

The Asymptotics Expansion to Derive the EQS and MQS Approximations



By electroquasistatic (EQS) approximation it is meant that the ordering of times is as to the left and that the parameter $\beta = (\tau_{em}/\tau)^2$ is much less than unity. Note that τ is still arbitrary relative to τ_e. In the magnetoquasistatic (MQS) approximation, β is still small, but the ordering of characteristic times is as to the right. In this case, τ is arbitrary relative to τ_m.

To make a formal statement of the procedure used to find the quasistatic approximation, the normalized fields and charge density are expanded in powers of the time-rate parameter β.

$$
\begin{aligned}
\vec{E} &= \vec{E}_o + \beta\vec{E}_1 + \beta^2\vec{E}_2 + \cdots \\
\vec{H} &= \vec{H}_o + \beta\vec{H}_1 + \beta^2\vec{H}_2 + \cdots \\
\vec{J}_v &= (\vec{J}_v)_o + \beta(\vec{J}_v)_1 + \beta^2(\vec{J}_v)_2 + \cdots \\
\rho_f &= (\rho_f)_o + \beta(\rho_f)_1 + \beta^2(\rho_f)_2 + \cdots
\end{aligned}
\tag{12}
$$

In the following, it is assumed that constitutive laws relate $\vec{P}$ and $\vec{M}$ to $\vec{E}$ and $\vec{H}$, so that these densities are similarly expanded. The velocity $\vec{v}$ is taken as given. Then, the series are substituted into Eqs. 5-9 and the resulting expressions arranged by factors multiplying ascending powers of β. The "zero order" equations are obtained by requiring that the coefficients of β^0 vanish. These are simply Eqs. 5-9 with $\beta = 0$:

$$
\begin{aligned}
&\nabla\cdot\vec{E}_o = -\nabla\cdot\vec{P}_o + (\rho_f)_o && \nabla\cdot\vec{E} = -\nabla\cdot\vec{P}_o + (\rho_f)_o && (13)\\
&\nabla\cdot\vec{H}_o = -\nabla\cdot\vec{M}_o && \nabla\cdot\vec{H}_o = -\nabla\cdot\vec{M}_o && (14)\\
&\nabla\times\vec{H}_o = \frac{\tau}{\tau_e}\sigma\vec{E}_o + (\vec{J}_v)_o + \frac{\partial\vec{E}_o}{\partial t} + \frac{\partial\vec{P}_o}{\partial t} + \nabla\times(\vec{P}_o\times\vec{v}) && \nabla\times\vec{H}_o = \frac{\tau_m}{\tau}\sigma\vec{E}_o + (\vec{J}_v)_o && (15)\\
&\nabla\times\vec{E}_o = 0 && \nabla\times\vec{E}_o = -\frac{\partial\vec{H}_o}{\partial t} - \frac{\partial\vec{M}_o}{\partial t} - \nabla\times(\vec{M}_o\times\vec{v}) && (16)\\
&\nabla\cdot\sigma\vec{E}_o + \frac{\tau_e}{\tau}\left[\nabla\cdot(\vec{J}_v)_o + \frac{\partial(\rho_f)_o}{\partial t}\right] = 0 && \nabla\cdot\sigma\vec{E}_o + \frac{\tau}{\tau_m}\nabla\cdot(\vec{J}_v)_o = 0 && (17)
\end{aligned}
$$

The zero-order solutions are found by solving these equations, augmented by appropriate boundary conditions. If the boundary conditions are themselves time dependent, normalization will turn up additional characteristic times that must be fitted into the hierarchy of Fig. 2.3.1.

Higher order contributions to the series of Eq. 12 follow from a sequential solution of the equations found by making coefficients of like powers of β vanish. The expressions resulting from setting the coefficients of β^n to zero are:

$$
\begin{aligned}
&\nabla\cdot\vec{E}_n + \nabla\cdot\vec{P}_n - (\rho_f)_n = 0 && \nabla\cdot\vec{E}_n + \nabla\cdot\vec{P}_n - (\rho_f)_n = 0 && (18)\\
&\nabla\cdot\vec{H}_n + \nabla\cdot\vec{M}_n = 0 && \nabla\cdot\vec{H}_n + \nabla\cdot\vec{M}_n = 0 && (19)\\
&\nabla\times\vec{H}_n - \frac{\tau}{\tau_e}\sigma\vec{E}_n - (\vec{J}_v)_n - \frac{\partial\vec{E}_n}{\partial t} - \frac{\partial\vec{P}_n}{\partial t} - \nabla\times(\vec{P}_n\times\vec{v}) = 0 && \nabla\times\vec{H}_n - \frac{\tau_m}{\tau}\sigma\vec{E}_n - (\vec{J}_v)_n = \left[\frac{\partial\vec{E}_{n-1}}{\partial t} + \frac{\partial\vec{P}_{n-1}}{\partial t} + \nabla\times(\vec{P}_{n-1}\times\vec{v})\right] && (20)\\
&\nabla\times\vec{E}_n = -\left[\frac{\partial\vec{H}_{n-1}}{\partial t} + \frac{\partial\vec{M}_{n-1}}{\partial t} + \nabla\times(\vec{M}_{n-1}\times\vec{v})\right] && \nabla\times\vec{E}_n + \frac{\partial\vec{H}_n}{\partial t} + \frac{\partial\vec{M}}{\partial t} + \nabla\times(\vec{M}_n\times\vec{v}) = 0 && (21)\\
&\nabla\cdot\sigma\vec{E}_n + \frac{\tau_e}{\tau}\left[\nabla\cdot(\vec{J}_v)_n + \frac{\partial(\rho_f)_n}{\partial t}\right] = 0 && \nabla\cdot\sigma\vec{E}_n + \frac{\tau}{\tau_m}\nabla\cdot(\vec{J}_v)_n = -\frac{\tau}{\tau_m}\frac{\partial(\rho_f)_{n-1}}{\partial t} && (22)
\end{aligned}
$$

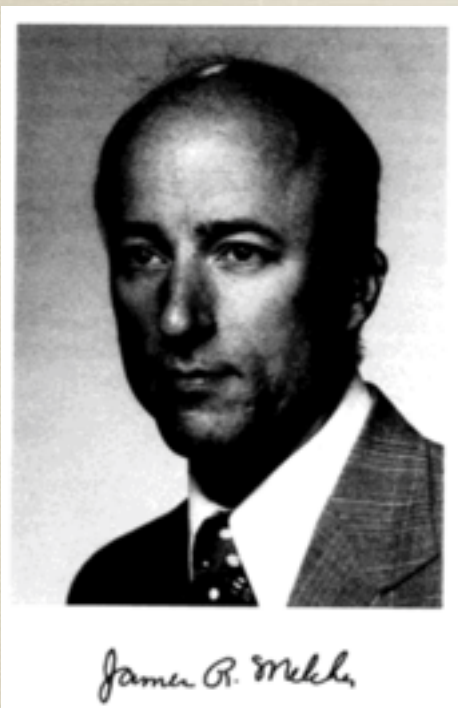
http://www.rle.mit.edu/cehv/documents/ContinuumElectromechanics-Melcher.pdf

The Fields and the Boundary Conditions in the Moving Frame

EQS	MQS	
$\vec E' = \vec E$	$\vec H' = \vec H$	(9)
$\vec P' = \vec P$	$\vec M' = \vec M$	(10)
$\rho_f' = \rho_f$	$\vec J_f' = \vec J_f$	(11)
$\vec J_f' = \vec J_f - \vec u \rho_f$	$\vec E' = \vec E + \vec u \times \mu_o \vec H$	(12)
$\vec H' = \vec H - \vec u \times \epsilon_o \vec E$		(13)
and hence, from Eq. 2.2.6	and hence, from Eq. 2.2.7	
$\vec D' = \vec D$	$\vec B' = \vec B$	(14)

The primary fields are the same whether viewed from one frame or the other. Thus, the EQS electric field polarization density and charge density are the same in both frames, as are the MQS magnetic field, magnetization density and current density. The respective dynamic laws can be associated with those field transformations that involve the relative velocity. That the free current density is altered by the relative motion of the net free charge in the EQS system is not surprising. But, it is the contribution of this same convection current to Ampère's law that generates the velocity dependent contribution to the EQS magnetic field measured in the moving frame of reference. Similarly, the velocity dependent contribution to the MQS electric field transformation is a direct consequence of Faraday's law.

The transformations, like the quasistatic laws from which they originate, are approximate. It would require Lorentz transformations to carry out a similar procedure for the exact electrodynamic laws of Sec. 2.2. The general laws are not invariant in form to a Galilean transformation, and therein is the origin of special relativity. Built in from the start in the quasistatic field laws is a self-consistency with other Galilean invariant laws describing mechanical continua that will be brought in in later chapters.



Summary of Electroquasistatic and Magnetoquasistatic Conditions: Table 2.10.1 summarizes the jump conditions.

Table 2.10.1. Quasistatic jump conditions; $[\![\vec A]\!] \equiv \vec A^a - \vec A^b$.

EQS	MQS	
$\vec n \cdot [\![\epsilon_o \vec E + \vec P]\!] = \sigma_f$ $\vec n \cdot [\![\vec P]\!] = -\sigma_p$	$\vec n \times [\![\vec H]\!] = \vec K_f$	(21)
$\epsilon_o [\![\Phi]\!] = \sigma_d$ $\epsilon_o [\![\vec E_t]\!] = -\nabla_\Sigma \sigma_d$	$\vec n \cdot \mu_o [\![\vec H + \vec M]\!] = 0$ $\vec n \cdot \mu_o [\![\vec M]\!] = -\sigma_m$	(22)
$\vec n \cdot [\![\vec J_f - \rho_f \vec v]\!] + \nabla_\Sigma \cdot \vec K_f = -\frac{\partial \sigma_f}{\partial t}$	$\vec n \times [\![\vec E + v \times \mu_o \vec H]\!] = 0$	(23)
$\vec n \times [\![\vec H - \vec v \times \epsilon_o \vec E]\!] = \vec K_f - \sigma_f \vec v_t$	$\vec n \cdot [\![\vec J_f]\!] = 0$	(24)

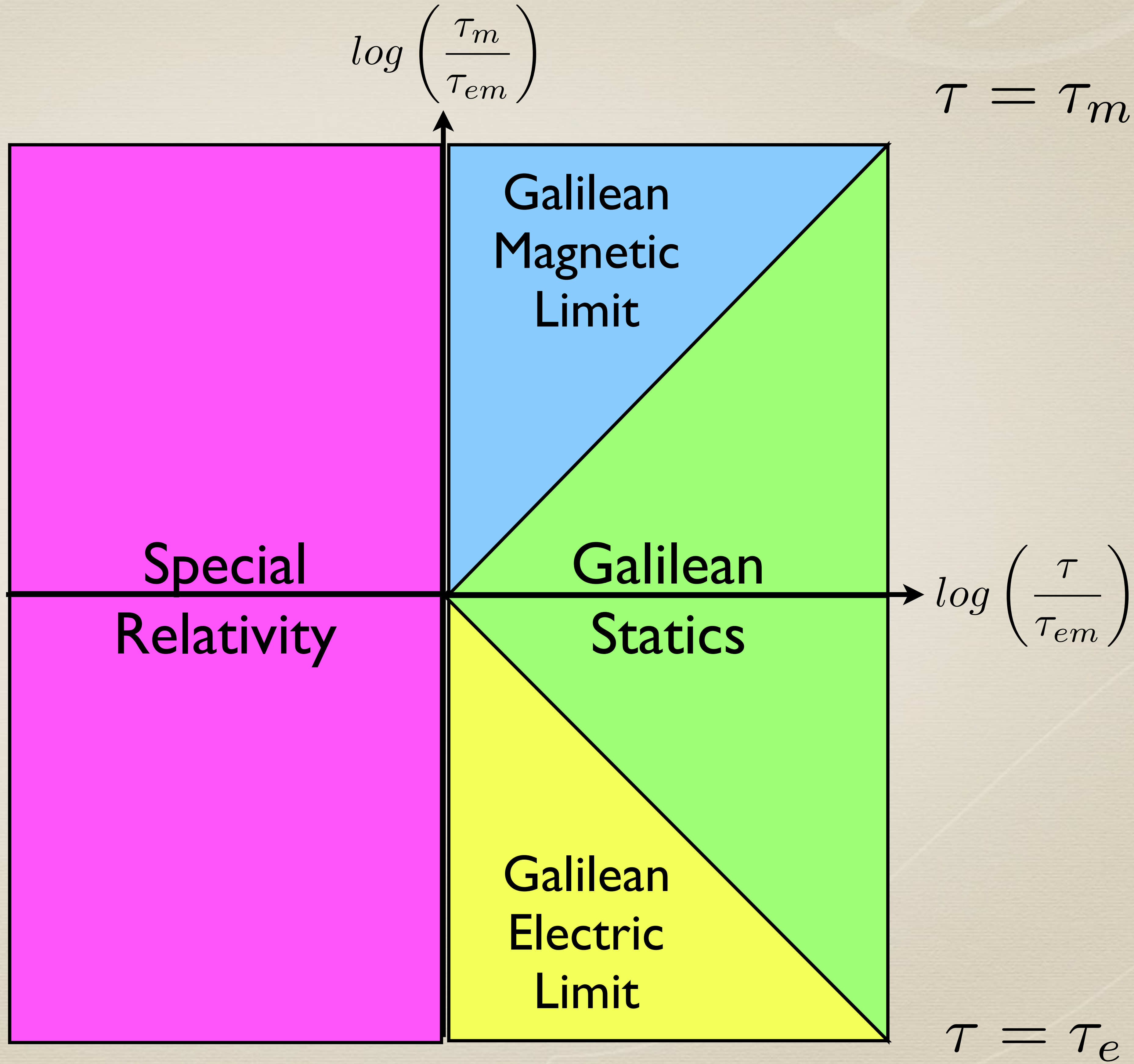
http://www.rle.mit.edu/cehv/documents/ContinuumElectromechanics-Melcher.pdf

The Range of Validity of the Galilean Limits of Electromagnetism

$$\tau_{em} = \frac{L}{c_m}$$

$$\tau_e = \sqrt{\frac{\epsilon}{\sigma}}$$

$$\tau_m = \mu\sigma L^2$$



EQS and MQS According to James Melcher

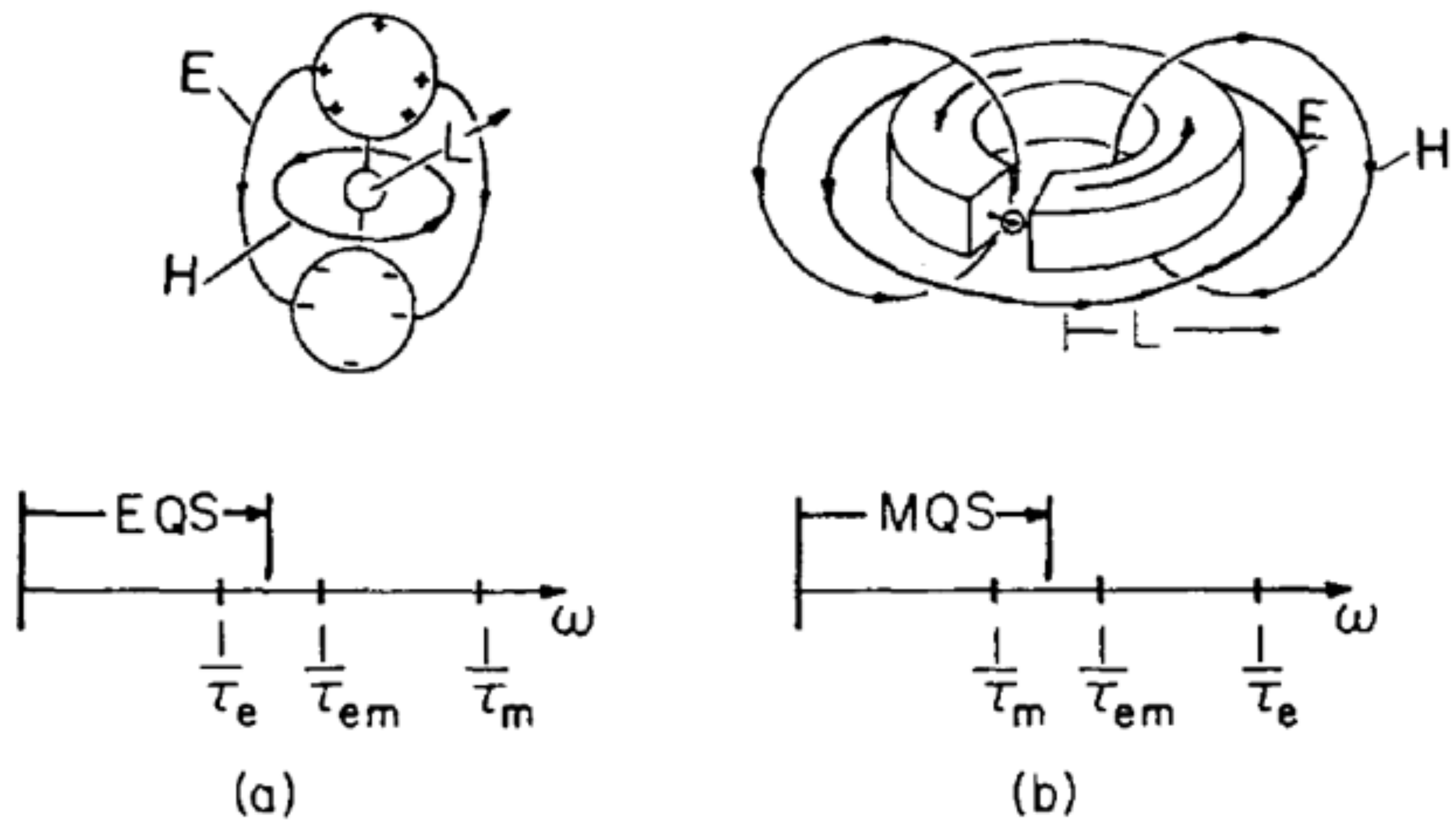


Fig. 2. As the angular frequency ω is raised, an electromagnetic system is first (a) EQS if $\tau_e > \tau_m$ and is (b) MQS if $\tau_m > \tau_e$. The representative EQS system in (a) has a voltage source driving a pair of perfectly conducting spheres having radius and spacing with the same typical length L. The representative MQS system in (b) has a perfectly conducting loop driven by a current source having radius and width with the same typical length L. For quasistatic systems, it is necessary that system dimensions be much smaller than the radiating wavelength ($L \ll \lambda$, $\lambda = 2\pi c/\omega$ and $c = [\varepsilon\mu]^{-1/2}$ is the speed of electromagnetic waves).

http://www.rle.mit.edu/cehv/documents/53-JournalofElectrostatics.pdf

Characteristics Size Scale and Times

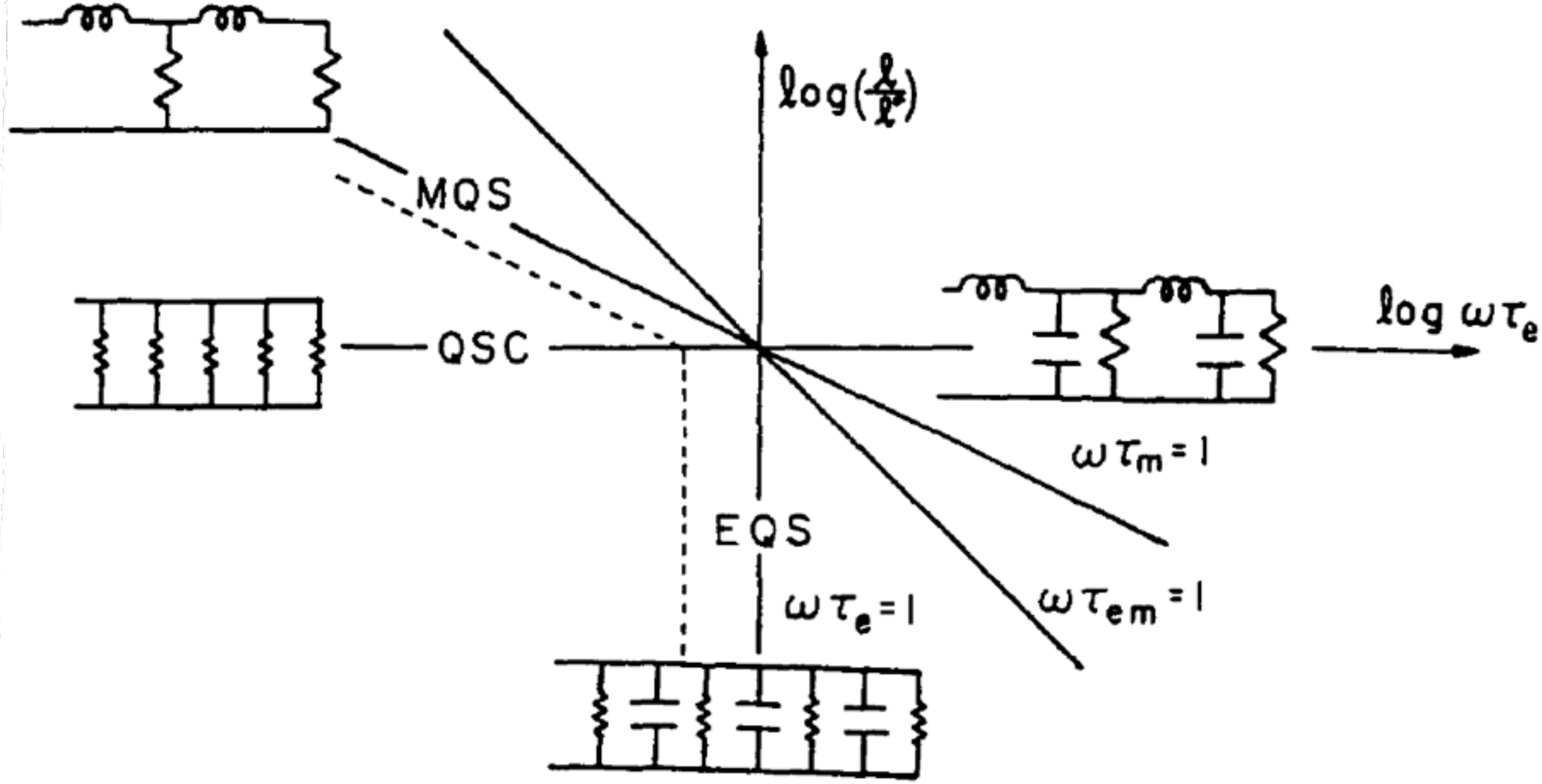


Fig. 3. With characteristic size scale $l^* = [\varepsilon/\mu]^{1/2}/\sigma$, systems are EQS if the system length scale $l < l^*$ and are MQS if $l > l^*$. The MQS and EQS regimes both become quasistationary conduction (QSC) at low frequencies such that both $\omega\tau_e \ll 1$ and $\omega\tau_m \ll 1$.

http://www.rle.mit.edu/cehv/documents/53-JournalofElectrostatics.pdf

The Vaschy-Buckingham Theorem of Dimensional Analysis

Parameter units in the MKSA system.

	μ	ϵ	σ	τ	ℓ
L	1	−3	−3	0	1
M	1	−1	−1	0	0
T	−2	4	3	1	0
I	−2	2	2	0	0

Three characteristic times, namely τ_{em}, τ_e and τ_m, appear as soon as we represent τ and ℓ in terms of the fundamental physical parameters ϵ, μ and σ. In the MKSA system for example, expressed in terms of mass M (Kg), length L (m), time T (s) and current I (A), the parameters' dimensions are

$$[\mu] = [L][M][T]^{-2}[I]^{-2},$$
$$[\epsilon] = [L]^{-3}[M]^{-1}[T]^{4}[I]^{2},$$
$$[\sigma] = [L]^{-3}[M]^{-1}[T]^{3}[I]^{2}.$$

Considering the numerical part of Table 1 as a 4×5 matrix, and remarking that the last two columns of the so-defined matrix contain just one non-zero unitary entry and that the last line is minus twice the second, the matrix rank is 3. Two parameters (τ and ℓ) can be expressed as functions of three others (μ, ϵ and σ). To this purpose, we seek for α, β, γ, c_1, c_2, and c_3 reals such that the following two ratios are dimensionless:

$$\tau/(\mu^{\alpha}\epsilon^{\beta}\sigma^{\gamma}) = O(1), \qquad \ell/(\mu^{c_1}\epsilon^{c_2}\sigma^{c_3}) = O(1).$$

The first ratio yields the following linear system

$$\begin{cases} \alpha - 3\beta - 3\gamma = 0, \\ \alpha - \beta - \gamma = 0, \\ -2\alpha + 4\beta + 3\gamma = 1, \\ -2\alpha + 2\beta + 2\gamma = 0 \end{cases}$$

whose solution is $\alpha = 0$, $\beta = 1$, and $\gamma = -1$ (the fourth equation coincides with the second one up to a multiplicative factor -2). We introduce the first quantity $\tau_e = \epsilon/\sigma$ and we have $\tau/\tau_e = O(1)$. Indeed, τ_e is the electric charge diffusion time that is the characteristic time during which the simple electric charge decays in a conductor.

For the second ratio, we have to find c_1, c_2, and c_3 solution of a similar linear system with right-hand side equal to $(1, 0, 0, 0)^t$. We thus get $c_1 = -1/2$, $c_2 = 1/2$, and $c_3 = -1$. We introduce $\ell^{\star} = (\sqrt{\epsilon/\mu})/\sigma$ and we have $\ell/\ell^{\star} = O(1)$. Since $\ell/\ell^{\star} = \mu\sigma\ell c = \mu\sigma\ell^2(c/\ell)$, a natural choice is to set $\tau_{em} = \ell/c$ and $\tau_m = \mu\sigma\ell^2$. The quantity τ_m is the current density

Les erreurs/oublis/impasses de 1973

- Dans la limite magnétique, les courants peuvent être instationnaires même si la densité de courant est irrotationnelle

- Dans la limite magnétique, une densité de courant vue d'un référentiel en mouvement est équivalent à une densité de charge

- Dans la limite électrique, les condensateurs marchent

- La force de Lorentz doit être définie dans le référentiel en mouvement

- Deux théorèmes de Poynting sont nécessaires

- Les relations constitutives doivent être généralisées

- Les limites ne dépendent pas du système d'unités

- c n'est pas la vitesse de la lumière mais le facteur de conversion d'unité

- Quid des conditions de jauges ?

Approche “Temporelle” dans les Milieux

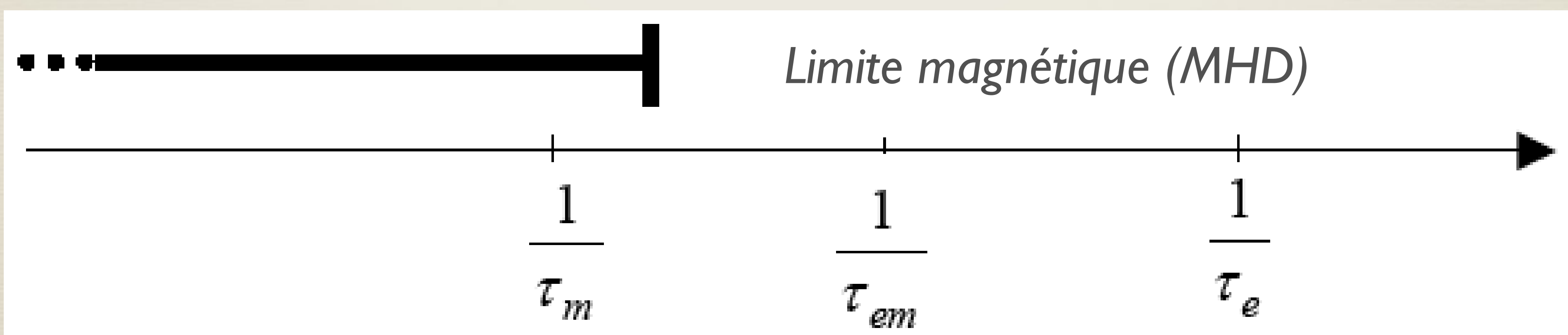
(Haus & Melcher, Moreau, Davidson, Fauve & Petrelis)

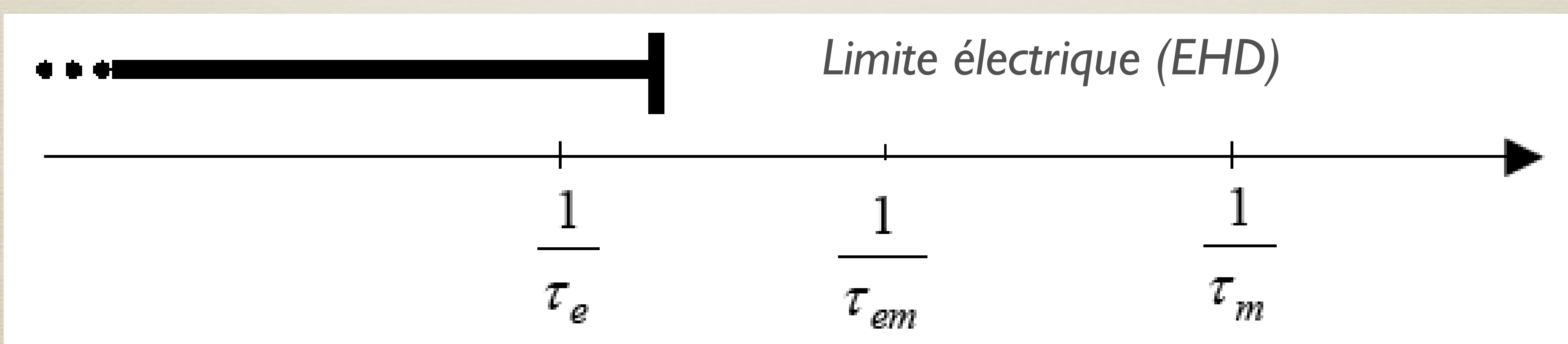
$$\tau_{em} = \frac{L}{c}$$

$$\tau_m = \mu_{ref}\ \sigma_{ref} L^2$$

$$\tau_e = \frac{\varepsilon_{ref}}{\sigma_{ref}}$$

$$\tau_{em}^2 = \tau_m\ \tau_e$$





Electrical equations

In Electrohydrodynamics Maxwell's equations reduce to the quasi-electrostatic equations:

$$\nabla \cdot \mathbf{D} = q, \qquad \nabla \times \mathbf{E} = 0, \qquad \frac{\partial q}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{1}$$

where q denotes the volume charge density.
For this to be true we must have

$$W_m = \frac{1}{2}\mu H^2 \ll W_e = \frac{1}{2}\epsilon E^2 \tag{2}$$

http://www.personal.soton.ac.uk/jss1x07/downloads/ehd_workshop/presentations/antonio/EHD-Southampton.pdf

Electrical equations (continuation)

From

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3}$$

the two following inequalities must be satisfied

$$J_0 \ll E_0 \sqrt{\frac{\epsilon}{\mu}} \frac{1}{l}, \qquad \frac{l\omega}{c} = 2\pi \frac{l}{\lambda} \ll 1 \tag{4}$$

For ohmic liquids, $J_0 = \sigma E_0$, the first inequality is $(\sigma \mu l^2)/(\sigma/\epsilon) = \tau_m/\tau_e \ll 1$.

Dimensional analysis

One considers field problems where

$E = e\mathcal{E}$, with e a reference quantity and $\mathcal{E}$ a non-dimensional vector (of order 1)

ℓ , a characteristic spatial dimension

τ , a characteristic time constant
the time interval in which significant changes of the field quantities arise

$v = \frac{\ell}{\tau}$ velocity of the phenomenon in the medium or of the medium

$c_m = \frac{1}{\sqrt{\mu\epsilon}}$ velocity of light in the medium (c in the empty space)

spatial differentiation $\partial_x E$ is approximated by $\dfrac{e}{\ell}$

time differentiation $\partial_t E$ is approximated by $\dfrac{e}{\tau}$

Galilean limits of Maxwell equations : for fields

In the empty space, two velocities and scalings appearing :

$$\nabla \times E = -\partial_t B \quad \rightarrow \quad \frac{e}{\ell} \sim \frac{b}{\tau} \quad \rightarrow \quad e \sim vb \qquad (v = \tfrac{\ell}{\tau})$$

$$\nabla \times H = \partial_t D \quad \rightarrow \quad \frac{b}{\mu_0 \ell} \sim \frac{\epsilon_0 e}{\tau} \quad \rightarrow \quad b \sim \tfrac{v}{c^2} e \qquad (c = \tfrac{1}{\sqrt{\mu_0 \epsilon_0}}).$$

$v \approx c$ yields $\begin{cases} e = cb \\ b = \dfrac{e}{c} \end{cases}$, and $v \ll c$ yields $\begin{cases} e = vb \\ b = \frac{v}{c^2} e \end{cases}$ which results in $v = c$, wrong !

This means that for $v \ll c$, the two scalings are not simultaneously valid
thus Faraday and Ampère laws cannot be coupled

To choose the scaling, look at the field sources (charges or currents) !

Galilean limits of Maxwell equations : for sources

$$\nabla \cdot D = \rho \quad \rightarrow \quad \frac{\epsilon_0 e}{\ell} \sim \rho \qquad \text{and} \qquad \nabla \times H = J \quad \rightarrow \quad \frac{b}{\mu_0 \ell} \sim J$$

$$\frac{\frac{b}{\mu_0 \ell}}{c\epsilon_0 \frac{e}{\ell}} \sim \frac{J}{\rho c} \quad \rightarrow \quad \frac{J}{\rho c} \sim \frac{b}{\mu_0 \epsilon_0 c e} \quad \rightarrow \quad \frac{J}{\rho c} \sim \frac{cb}{e}$$

if $J \gg \rho c$ then $e \ll cb$, thus MQS (the dielectric effect of charges is negligible)

if $J \ll \rho c$, then $e \gg cb$, thus EQS (the conducting effect is negligible)

Galilean limits of Maxwell equations : for potentials

$A = a\mathcal{A}$ (magnetic vector potential), $V = \mathtt{V}\mathcal{V}$ (electric scalar potential)

$B = \nabla \times A$ yields $b \sim \frac{a}{\ell}$ (always true)

$E = -\partial_t A - \nabla V$ yields $e \sim \frac{a}{\tau} + \frac{\mathtt{V}}{\ell}$ thus

- if $c\,a \ll \mathtt{V}$ then

$$(v \ll c) \qquad v\,a \ll \mathtt{V}, \qquad \frac{\ell}{\tau} a \ll \mathtt{V}, \qquad \frac{a}{\tau} \ll \frac{\mathtt{V}}{\ell}$$

so $e \sim \frac{\mathtt{V}}{\ell}$, thus $E = -\nabla V$ and $\nabla \times E = 0$ (EQS).

- if $c\,a \gg \mathtt{V}$ then

$$a \gg \mathtt{V} \text{ is compensated by } v \ll c, \qquad \frac{a}{\tau} \sim \frac{\mathtt{V}}{\ell}$$

so $E = -\partial_t A - \nabla V$ that yields $\nabla \times E = -\partial_t B$ (MQS).

Galilean limits of Maxwell equations : for gauge condition

$A = a\mathcal{A}$ (magnetic vector potential), $V = \mathtt{V}\mathcal{V}$ (electric scalar potential)

$$\nabla \cdot A + \frac{1}{c^2}\partial_t V = 0, \qquad \frac{a}{\ell} + \frac{\mathtt{V}}{c^2\tau} \sim 0, \qquad c\,a + \frac{v\,\mathtt{V}}{c} \sim 0$$

- $c\,a \gg \mathtt{V}$ (MQS) then $a \gg \mathtt{V}$ yields $\nabla \cdot A = 0$ Coulomb gauge condition

- $c\,a \ll \mathtt{V}$ (EQS) plus $\frac{v}{c} \ll 1$ yields Lorenz gauge condition

$b \sim \frac{a}{\ell}$ (always true) and $\begin{cases} e \sim \frac{a}{\tau} \sim \frac{\mathtt{V}}{\ell} & (MQS) \\ e \sim \frac{\mathtt{V}}{\ell} \gg \frac{a}{\tau} & (EQS). \end{cases}$

$$\frac{cb}{e} \sim \frac{c\frac{a}{\ell}}{\frac{a}{\tau}} \sim \frac{c}{v} \gg 1 \qquad (MQS)$$

Lorenz gauge condition on potentials, $\nabla \cdot A + \frac{1}{c^2}\partial_t V = 0$, implies $\frac{a}{\ell} \sim \frac{\mathtt{V}}{c^2\tau}$

$$\frac{cb}{e} \sim \frac{c\frac{a}{\ell}}{\frac{\mathtt{V}}{\ell}} \sim \frac{c\frac{a}{\ell}}{\frac{c^2 a}{\ell v}} = \frac{v}{c} \ll 1 \qquad (EQS)$$

Stratton gauge condition (1941)

$$\nabla \cdot A + \mu\epsilon\partial_t V = -\mu\sigma V, \qquad \frac{a}{\ell}\nabla' \cdot \mathcal{A} + \mu\epsilon\frac{\mathtt{V}}{\tau}\partial_{t'}\mathcal{V} = -\mu\sigma\mathtt{V}\mathcal{V}$$

$$\nabla' \cdot \mathcal{A} + \frac{\frac{\ell}{\tau}}{c_m}\frac{\mathtt{V}}{c_m\, a}\partial_{t'}\mathcal{V} = -\frac{\ell}{\ell^*}\frac{\mathtt{V}}{c_m\, a}\mathcal{V}, \qquad I + II = III$$

Dimensionless ratios

$$\frac{II}{I} \sim \frac{v}{c_m}\frac{\mathtt{V}}{c_m a} = \frac{\tau_{em}}{\tau}\frac{\mathtt{V}}{c_m a}$$

$$\frac{III}{I} \sim \frac{\ell}{\ell^*}\frac{\mathtt{V}}{c_m a} = \frac{\tau_m}{\tau_{em}}\frac{\mathtt{V}}{c_m a} = \frac{\tau_{em}}{\tau_e}\frac{\mathtt{V}}{c_m a}$$

$$\frac{III}{II} \sim \frac{\tau}{\tau_e}$$

$\ell^* = \frac{1}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$ is the constitutive length

$\tau_{em} = \frac{\ell}{c_m}$ the light transit time in the medium

$\tau_e = \frac{\epsilon}{\sigma}$ the charge relaxation time

$\tau_m = \mu\sigma\ell^2$ the magnetic diffution time such that $\tau_{em} = \sqrt{\tau_e\tau_m}$.

Domains of validity

MQS : $0 \overline{\qquad\qquad} \tau_e \overline{\qquad\qquad} \tau_{em} \overline{\qquad\qquad} \tau_m \overline{\qquad\qquad} \tau$

EQS : $0 \overline{\qquad\qquad} \tau_m \overline{\qquad\qquad} \tau_{em} \overline{\qquad\qquad} \tau_e \overline{\qquad\qquad} \tau$

$\frac{\tau_e}{\tau_{em}} = \frac{\ell^*}{\ell}, \qquad \frac{\tau_m}{\tau_{em}} = \frac{\ell}{\ell^*}, \qquad x := \log(\frac{\tau}{\tau_{em}}), \qquad y := \log(\frac{\ell}{\ell^*})$